Algebra Qualifying Examination
August 2022

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. "Rough working" will not receive credit: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but may not be used as a step in your reasoning).
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.

Notation: Throughout, $\mathbb{Z}$ denotes the ring of integers $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p$ denote respectively the fields of rational, real, and complex numbers, and a prime field of characteristic $p \neq 0$ for a prime p (i.e., a field with p elements). Also $M_n(R)$ denotes the ring of matrices with entries in the ring R.

1. (10) Show that if the center of a group G has index n in G, then every conjugacy class in G has at most n elements.
2. (15) Let $\mathrm{GL}_3(\mathbb{F}_3)$ be the group of invertible 3×3 matrices with entries in the field with 3 elements. Suppose that G is its subgroup of upper triangular matrices with 1s on the diagonal,
$$G \;=\; \left\{ \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) \text{ such that } x, y, z \in \mathbb{F}_3 \right\} \;.$$
Determine the order of G and whether or not G is abelian. Find the center of G.
3. (15) A left ideal I in a ring R is said to be regular if there exists $e \in R$ such that $r - re \in I$ for every $r \in R$. Show that if $I \neq R$ is a regular left ideal of a ring R, then I is contained in a maximal left ideal which is regular. (Observe that it is not assumed that R has an identity element).
4. (10) Find the smallest nonnegative integer $c \geq 0$ for which $R_c := \mathbb{Z}[x]/\langle c, x^2 - 2\rangle$ is
   (a) a domain.
   (b) a field.

5. (10) Let D be a division ring. Show that $M_n(D)$ is a simple ring (i.e. every proper ideal is zero).

6. (10) Suppose I is a principal ideal in the integral domain R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements (i.e., $rm = 0$ with $0 \neq r \in R$ and $m \in I \otimes_R I$ implies that $m = 0$).

7. (15) Suppose that k is a field whose characteristic is not 2 or 3, and let t be an indeterminate. Consider the (separable) polynomial $f(x) = x^3 - t \in k(t)[x]$ over the field $k(t)$. The Galois group of f may depend on k. Describe the Galois group for $k = \mathbb{Q}, \mathbb{R}, \mathbb{F}_5, \mathbb{F}_7, \mathbb{C}$.

8. (15) Suppose p is a prime number and f is a non-constant polynomial in $\mathbb{F}_p[x]$.

   (a) Prove that taking pth powers defines a linear endomorphism M_f on $\mathbb{F}_p[x]/\langle f \rangle$.

   (b) Pick an $\mathbb{F}_3$-vector space basis for $R := \mathbb{F}_3[x]/\langle x^3 + 2x + 1 \rangle$ and, relative to this basis, find an explicit matrix that encodes the linear map defined by taking 3rd powers.

   (c) Suppose f is a square-free (i.e not divisible by the square of any non-constant polynomial). Prove that f is irreducible if and only if $M_f - Id$ has kernel of dimension 1 where Id is the identity endomorphism.