Geometry/Topology Qualifying Exam — Spring 2024
Texas A&M University — Thursday, 11 January 2024

- Justify all your assertions.
- There are 10 problems. Try to solve all of them and make solutions and proofs as complete as possible.
- Use a separate sheet for each problem.
- Write your name on the top right corner of each page.

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1. Let X and Y be topological spaces, and let $\pi_X \colon X \times Y \to X$ be the projection on the first coordinate, that is, $\pi_X(x, y) = x$ for $(x, y) \in X \times Y$. Prove or disprove the following assertions:
   (a) π_X is a continuous map.
   (b) π_X is an open map.
   (c) π_X is a closed map.
   (d) π_X is a quotient map.

2. The branching line B is the topological space obtained as the quotient space of $\mathbb{R} \times \{0, 1\}$ with respect to the equivalence relation $(x, 0) \sim (x, 1)$ if and only if $x < 0$. Prove or disprove the following assertions:
   (a) B is path-connected.
   (b) B is locally compact, that is, every point has a neighborhood which is itself contained in a compact set.
   (c) B is Hausdorff.
   (d) B is a T_1 space, that is, for every pair of distinct points p and $q \in B$, there exist a neighborhood U_p of p and a neighborhood U_q of q such that $q \notin U_p$ and $p \notin U_q$.
   (e) B is second-countable.

3. Let (X, d) be a metric space, and let Y be a non-empty subset of X. Let $f \colon X \to \mathbb{R}_{\geq 0}$ be the distance function from Y, that is,
$$f(x) = \inf \{d(x, y) \mid y \in Y\}.$$
   Show that $f(x) = 0$ if and only if $x \in \bar{Y}$, where $\bar{Y}$ denotes the closure of Y.

4. Let $p \colon E \to B$ be a covering space. Fix a basepoint $b_0 \in B$, and suppose $p^{-1}(b_0)$ has k elements.
   (a) Assume B is connected. Show that $p^{-1}(b)$ has also k elements, for every $b \in B$. Prove the assertion under the assumption that B is path-connected to get half points.
   (b) Assume B is compact. Show that E is also compact.

5. (a) Compute the fundamental group of the 2-sphere with k points removed.

(b) Let $\ell_1, \ldots, \ell_n$ be n distinct lines in $\mathbb{R}^3$ passing through the origin. Let L be the union of these lines, that is, $L = \bigcup_{i=1}^n \ell_i$. Compute the fundamental group of $\mathbb{R}^3 \backslash L$.

6. (a) Formulate the implicit function theorem (you do not have to prove it).

   (b) Let n be a positive integer and let $O(n)$ denote the set of orthogonal $n \times n$ matrices as a subset of the set of all $n \times n$ matrices $M(n,n)$ (which can be identified with the Euclidean space $\mathbb{R}^{n^2}$). Prove that $O(n)$ is an embedded submanifold of $M(n,n)$ and find its dimension.

7. Let M and N be smooth manifolds and let $f : M \to N$ be a smooth map.

   (a) Define the map $f^* : \Omega^k(N) \to \Omega^k(M)$ that pulls k-forms on N back to k-forms on M.

   (b) For a 1-form $\omega \in \Omega^1(N)$, show that

$$d\left(f^*\omega\right) = f^*\left(d\omega\right).$$

8. Consider the plane $\mathbb{R}^2$ (with coordinates (x,y)) equipped with the metric

$$\frac{4}{(1+x^2+y^2)^2}\left(dx^2 + dy^2\right).$$

   Find the Gaussian curvature of this metric at each point.

9. Equip the Euclidean space $\mathbb{R}^3$ with cylindrical coordinates (r,θ,z) (so that $x = r\cos\theta$, $y = r\sin\theta$, $z = z$). Let Δ be the distribution spanned by X and Y, where

$$X = \frac{\partial}{\partial r}, \quad \text{and} \quad Y = \frac{\partial}{\partial \theta} - r^2 \frac{\partial}{\partial z}.$$

   Is the distribution Δ integrable?

10. Let ω be a closed 1-form (so $d\omega = 0$) on a smooth manifold M. Prove that ω is exact (so $\omega = df$ for some smooth function f on M) if and only if

$$\int_\gamma \omega = 0$$

   for every smooth closed curve γ on M.