Qualifying exam — August 8, 2023

1. Prove that the group of all invertible upper triangular matrices of size $n \times n$ over a field is solvable.

2. Let $\mathbb{F}_3$ be the field of 3 elements. Consider the group $\mathrm{PSL}_2(\mathbb{F}_3)$ defined as the quotient of the group of 2×2 matrices over $\mathbb{F}_3$ of determinant equal to 1 by the subgroup of diagonal matrices.

a) Find the order of $\mathrm{PSL}_2(\mathbb{F}_3)$.

b) Prove that $\mathrm{PLS}_2(\mathbb{F}_3)$ is isomorphic to A_4 (hint: consider lines in the plane $\mathbb{F}_3^2$).

3. Let G be a group of order 2000 with more than one Sylow 5-subgroup.

a) Show that any Sylow 5 sugroup P of G satisfies $P = N_G(P)$, i.e., $g^{-1}Pg = P$ for $g \in G$ implies $g \in P$.

b) Prove that if P, P' are two distinct Sylow 5-subgroups of G, then $|P \cap P'| = 25$.

4. Let D be an associative ring with unit without zero divisors. Suppose that there exists a subring K of D such that $ax = xa$ for all $a \in K$ and $x \in D$, K is a field (with respect to the operations of D), and D is finite-dimensional as a vector space over K. Show that D is a division ring (i.e., every non-zero element is invertible).

5. Let R be the ring of 2×2 matrices over the ring $\mathbb{Z}/6\mathbb{Z}$. Describe all two-sided ideals of R.

6. Let I and J be two distinct maximal ideals in a commutative ring R.

a) Prove that one has an exact sequence

$$0 \to IJ \to I \oplus J \to R \to 0$$

of R-modules.

b) Prove that if, in addition, I and J are projective R-modules, then IJ is a projective R-module.

7. Let ζ be a primitive 7th root of unity in $\mathbb{C}$. Let $\alpha = \zeta + \zeta^{-1}$.

a) Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

b) Show that if $\beta \in \mathbb{Q}(\alpha)$ satisfies $\beta^3 \in \mathbb{Q}$, then $\beta \in \mathbb{Q}$.

8. Find all automorphisms of the fields $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{2} + \sqrt{3})$, and $\mathbb{Q}(\sqrt[3]{2})$.