Algebra Qualifying Examination
August 9, 2017

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed next to each problem number.
- Credit awarded for your answers will be based on both the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and must be legible.
- Please do not interpret a problem in a way that makes it trivial.
- You may use a calculator to check your computations (but it may not be used as a step in your reasoning).
- Start each problem on a new page, and mark clearly the problem number and your name on each page turned in.

Notation: Throughout, let $\mathbb{Z}$ denote the ring of integers; let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively. All rings are assumed to have a multiplicative identity. For a Galois extension of fields L/K, let $\mathrm{Gal}(L/K)$ denote its Galois group.

1. [12 points] Let p and q be distinct prime numbers, and let G be a group of order p^2q. Prove that G has a normal Sylow subgroup. (Note: the cases $p > q$ and $p < q$ should be treated separately.)

2. [12 points; (a) 3 pts., (b) 6 pts., (c) 3 pts.] Let G be a group of order $375 = 3 \cdot 5^3$. Let H be a Sylow 3-subgroup, let K be a Sylow 5-subgroup, and suppose that K is cyclic.

   (a) According to the Sylow theorems, what are the possible numbers of conjugates of H? And of K?

   (b) Let $X = \{g \in K \mid K = \langle g \rangle\}$. Show that H acts on X by conjugation. What are the possible sizes of orbits of this action? What is the size of the set of generators of K?

   (c) Prove that G must be cyclic.

3. [12 points] Let K be a field. Prove that the polynomial ring $K[x]$ has infinitely many maximal ideals.

4. [12 points] Let V be a finite dimensional vector space over $\mathbb{C}$, and suppose we have $\mathbb{C}$-linear maps $A_1, \ldots, A_k : V \to V$ such that for all i, j, we have $A_i \circ A_j = A_j \circ A_i$. Show that there exists a non-zero vector in V that is simultaneously an eigenvector for each of $A_1, \ldots, A_k$ (with possibly different eigenvalues). (Suggestion: use an induction on k.)

5. [12 points; (a) 5 pts., (b) 7 pts.] Let R be a commutative ring, and let I, $J \subseteq R$ be ideals. Let $\varphi : R \to R/I \otimes_R R/J$ be the function defined by

$$\varphi(r) = r(\overline{1} \otimes \overline{1}), \qquad \forall\, r \in R.$$

(Here we are letting $\overline{1}$ denote either the coset $1 + I$ or $1 + J$, depending on the context.)

(a) Prove that φ is a surjective R-module homomorphism.

(b) Prove that the kernel of φ is $I + J$. (Hint: First show there is an R-module map $\psi : R/I \otimes_R R/J \to R/(I + J)$ such that $\psi(\overline{x} \otimes \overline{y}) = \overline{xy}$. Here as above, $\overline{x} = x + I$, $\overline{y} = y + J$, and $\overline{xy} = xy + (I + J)$.)

6. [12 points; (a) 5 pts., (b) 7 pts.] Let R be a commutative ring, let P and F be left R-modules, and let

$$\mathrm{Hom}_R(P, F) = \{f : P \to F \mid f \text{ is an } R\text{-module homomorphism}\}.$$

(a) For $r \in R$ and $f \in \mathrm{Hom}_R(P, F)$, show that the function $rf : P \to F$ defined by $(rf)(x) := f(rx)$, for $x \in P$, is an R-module homomorphism. Show that this makes $\mathrm{Hom}_R(P, F)$ into a well-defined left R-module.

(b) Assume further that both P and F are finitely generated as R-modules, that P is a projective R-module, and that F is a free R-module. Prove that $\mathrm{Hom}_R(P, F)$ is a projective R-module.

7. [12 points] Let $\alpha = \sqrt{1 + \sqrt{2}} \in \mathbb{R}$.

(a) What is the irreducible polynomial of α over $\mathbb{Q}$?

(b) Prove that $\mathbb{Q}(\alpha)$ is not the splitting field over $\mathbb{Q}$ of any polynomial in $\mathbb{Q}[x]$.

8. [16 points; (a) 3 pts., (b) 4 pts., (c)–(e) 3pts. each] Let $f = x^3 - 2 \in \mathbb{Q}[x]$, and let $g = x^2 - 2 \in \mathbb{Q}[x]$. Let K, L, and M be subfields of $\mathbb{C}$ such that K is the splitting field of f, L is the splitting field of g, and M is the splitting field of fg.

(a) Construct an automorphism $\beta \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\beta(\sqrt[3]{2}) = \omega\sqrt[3]{2}$ and $\beta(\omega) = \omega^2$, where $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$ is a cube root of unity.

(b) What is the order of β in $\mathrm{Gal}(K/\mathbb{Q})$? What is the fixed field of the subgroup generated by β?

(c) Determine $[M : \mathbb{Q}]$.

(d) Construct an element $\rho \in \mathrm{Gal}(M/\mathbb{Q})$ that has order 6, and determine its action on the roots of fg.

(e) What is the fixed field of the subgroup generated by the element ρ you constructed in (d)?