Algebra Qualifying Examination
7 January 2014

Instructions:

- There are eight questions worth a total of 100 points. Individual point values are indicated with each problem number.
- Read all problems first; make sure that you understand them and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit is awarded based both on the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and be legible. Do 'scratch work' on a separate page.
- Start each problem on a new page, clearly marking the problem number and your name on that page.
- Rings always have an identity (otherwise they are rng) and all R-modules are left modules.

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1. [12] A subgroup H of a group G is *characteristic* if $\varphi(H) = H$ for any automorphism φ of G. Show that a characteristic subgroup is normal. Suppose that $G = HK$, where H and K are characteristic subgroups of G with $H \cap K = \{e\}$. Show that $\mathrm{Aut}(G) \simeq \mathrm{Aut}(H) \times \mathrm{Aut}(K)$. (Here, $\mathrm{Aut}(L)$ is the group of automorphisms of L.)

2. [12] Show that any group of order $2014 = 2 \cdot 19 \cdot 53$ has a normal cyclic subgroup of index 2. Use this to classify all groups of order 2014.

3. [10] Prove that a finite integral domain is a field. Prove that every prime ideal in a finite commutative ring is maximal.

4. [14] Let R be a commutative ring. Observe that for any two R-modules M, N, the collection $\mathrm{Hom}(M, N)$ of R-module homomorphisms $\varphi\colon M \to N$ is naturally an R-module. Suppose that

$$0 \longrightarrow L \xrightarrow{e} M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

is an exact sequence of R-modules (so that g is a surjection whose kernel is equal to the image $f(M)$ of M under f, and e is an injection whose image is the kernel of f). Let A be an R-module. Prove that the induced sequence

$$0 \longrightarrow \mathrm{Hom}(A, L) \xrightarrow{e_*} \mathrm{Hom}(A, M) \xrightarrow{f_*} \mathrm{Hom}(A, N)$$

is exact in that e_* is injective and its image is the kernel of the map f_*. Also prove that the induced sequence

$$\mathrm{Hom}(M, A) \xleftarrow{f^*} \mathrm{Hom}(N, A) \xleftarrow{g^*} \mathrm{Hom}(P, A) \longleftarrow 0$$

is exact in that g^* is injective and its image is the kernel of the map f^*.

5. [10] Let M be an invertible $n \times n$ matrix with real number entries and positive determinant. Show that M can be written as RK where R is in $SO(n)$ (R is orthogonal with determinant 1) and K is an upper triangular matrix with positive entries on the diagonal. Hint: Orthogonal matrices have orthonormal column vectors.

6. [16] Consider a finite field $\mathbb{F}$ with $q = p^n$ elements, where p is a prime number and n is a positive integer.
   (a) Explain why every element of $\mathbb{F}$ is a root of the polynomial $x^{p^n} - x$.
   (b) Show that if r divides $p^n - 1$ then all the roots of the polynomial $x^r - 1$ of lie in $\mathbb{F}$.
   (c) Show that the polynomial $x^4 + 1$ is reducible over any finite field. (Hint: It is enough to show it over the prime fields with p elements. Consider the cases $p = 2$ and p odd separately and observe that for p odd, $p^2 - 1$ is congruent to 0 mod 8, and $x^8 - 1 = (x^4 - 1)(x^4 + 1)$.)

7. [14] Let $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$, let $\mathbb{E}$ be its splitting field contained in $\mathbb{C}$, and let G be the Galois group of $\mathbb{E}$ over $\mathbb{Q}$. Without simply citing a theorem about Galois groups of quartic polynomials, prove that G is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Find a generator for G and determine how it acts on the roots of $f(x)$. It may help to first identify an intermediate subfield $\mathbb{F}$, where $\mathbb{Q} \subsetneq \mathbb{F} \subsetneq \mathbb{E}$.

8. [12] Let p and q be prime numbers.
   (a) Define a surjective map $\phi : \mathbb{Q}(\sqrt{p}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{q}) \to \mathbb{Q}(\sqrt{p}, \sqrt{q})$ that is both $\mathbb{Q}$-linear and a ring homomorphism.
   (b) If p and q are distinct, show that ϕ is an isomorphism.
   (c) If $p = q$, what is a $\mathbb{Q}$-basis for the kernel of ϕ?