Applied Analysis Part

January 11, 2022

Name: ____

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let ψ_j and ϕ_j, $j = 1, \dots, n$, be in $L^2[0,1]$. Assume the sets $\{\psi_j\}_{j=1}^n$ and $\{\phi_j\}_{j=1}^n$ are linearly independent. Consider the finite rank kernel $k(x,y) = \sum_{j=1}^n \psi_j(x)\overline{\phi}_j(y)$ and let $Ku(x) = \int_0^1 k(x,y)u(y)dy$. You are given that K is compact.

(a) State and prove the Fredholm Alternative.
(b) State the Closed Range Theorem.
(c) Show that the equation $(I - \lambda K)u = f$ has an L^2-solution for all $f \in L^2[0,1]$ if and only if $1/\bar{\lambda}$ is *not* an eigenvalue of the matrix A, where $A_{jk} = \langle \phi_j, \psi_k \rangle$.

Problem 2. Let both $K \in \mathcal{C}(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{H})$ be self adjoint.

(a) Show that $\|L\|_{op} = \sup_{\|u\|=1} |\langle Lu, u\rangle|$. (Hint: look at $\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle$, then apply the polarization identity.)
(b) Prove this: *Either* $\|K\|$ *or* $-\|K\|$ *is an eigenvalue of* K.
(c) Let $\mathcal{H} = L^2[01]$ and define the operator $M : L^2[0,1] \to L^2[0,1]$ by $Mu(x) = xu(x)$. Show that $\|M\|_{op} = 1$. Is M compact? Prove your answer.

Problem 3. Suppose that $Lu = u'' + \lambda u$, with $\mathrm{Dom}(L) = \{u \in L^2(-\infty,\infty) : Lu \in L^2(-\infty,\infty)\}$, where $\lambda \in \mathbb{C} \setminus [0,\infty)$. In addition, choose $\mathrm{Im}\sqrt{\lambda} > 0$. Show that the Green's function for L is given by

$$g(x,y,\lambda) = \tfrac{-i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x-y|}$$

Problem 4. Consider the functions ϕ and ψ defined below:

$$\phi(x) = \begin{cases} (|x|-1)^2(2|x|+1) & |x| \le 1 \\ 0 & |x| > 1 \end{cases}, \qquad \psi(x) = \begin{cases} x(|x|-1)^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}.$$

Recall that for $n \ge 2$ and $0 \le j \le n$, the functions $\phi_j(x) := \phi(nx - j)$ and $\psi_j(x) := \frac{1}{n}\psi(nx - j)$ satisfy $\phi_j(k/n) = \delta_{j,k}$, $\phi_j'(k/n) = 0$, $\psi_j(k/n) = 0$ and $\psi_j'(k/n) = \delta_{j,k}$. In addition, the set $\{\phi_j, \psi_j\}_{j=0}^n$ is a basis for the finite element space $S^{\frac{1}{n}}(3,1)$.

(a) Let $S_0^{1/n}(3,1) = \{s \in S^{\frac{1}{n}}(3,1) : s(0) = s(1) = 0\}$. Show that $\langle u, v\rangle = \int_0^1 u''v''dx$ defines an inner product on $S_0^{1/n}(3,1)$, and that $\{\phi_j\}_{j=1}^{n-1} \cup \{\psi_j\}_{j=0}^n$ is a basis for $S_0^{1/n}(3,1)$.
(b) Show that $\langle \psi_j, \psi_k \rangle = 0$ for all j,k such that $|j-k| > 1$.
(c) Show that $\mathrm{argmin}\{\|s\| : s \in S_0^{\frac{1}{n}}, s(j/n) = f_j,\ j = 1, \dots n-1\}$ is given by $s(x) = \sum_{j=1}^{n-1} f_j\phi_j(x) - \sum_{j=0}^n \alpha_j\psi_j(x)$, where α_j's satisfy a tridiagonal system. Why is this system invertible?

NUMERICAL ANALYSIS QUALIFIER

January 11, 2022

Name: ______________________________

Problem 1. Let Ω be a polygonal domain in $\mathbf{R}^2$ and assume that $0 \in \Omega$. Let $u \in H^1(\Omega)$ be the solution of

$$a(u, \varphi) \ = \ l(\varphi), \qquad \text{for all} \ \ \varphi \in H^1(\Omega), \tag{1.1}$$

where the bilinear form $a(.\,,.)$ and, for a given function $f \in L^2(\Omega)$, the right hand side $l(\,.\,)$ are defined as follows,

$$a(v, \varphi) \ := \ \int_\Omega (\nabla v \cdot \nabla \varphi \ + \|x\|^2 \, v \, \varphi) dx, \qquad l(\varphi) \ := \ \int_\Omega f \varphi dx \qquad \text{for all} \ \ v, \varphi \in H^1(\Omega).$$

Let $\mathcal{T}_h$, $0 < h < 1$, be a familiy of shape regular triangulations of Ω. The elements of these partitions will be denoted by T_h. Set

$$V_h := \{v_h \in H^1(\Omega) \ : \ v_h|_{T_h} \in \mathcal{P}^1, \quad T_h \in \mathcal{T}_h\},$$

where $\mathcal{P}^1$ denotes the space of polynomials on $\mathbf{R}^2$ of degree at most 1.

(a) Show that the bilinear form $a(.\,,.)$ is coercive in $H^1(\Omega)$.
Hint: Decompose Ω into $\Omega_i := \Omega \cap B_\varepsilon(0)$ and $\Omega_o := \Omega \setminus \Omega_i$, where $B_\varepsilon(0)$ is a disc with radius ε around 0 such that $B_\varepsilon(0) \subset \Omega$. **You can use without proof that**

$$\|v\|^2_{L^2(\Omega)} \ \leq \ C \left\{ \|v\|^2_{L^2(\Omega_o)} \ + \ \|\nabla v\|^2_{L^2(\Omega)^2} \right\}$$

for all $v \in H^1(\Omega)$, for a suitable constant $C \geq 1$ depending on ε but independent of v.

(b) Given that $a(\cdot,\cdot)$ and $l(\cdot)$ are continuous, there exists a unique weak solution $u \in H^1(\Omega)$ of (1.1). Derive the strong form of problem (1.1) assuming that the solution u is smooth.

Now, consider the following finite element ansatz: find $u_h \in V_h$ such that

$$a(u_h, \varphi_h) \ = \ l(\varphi_h), \quad \forall \varphi_h \in V_h. \tag{1.2}$$

(c) State and prove Cea's Lemma for the error of the FE solution in the H^1-norm.

(d) Assuming that the solution u is in $H^2(\Omega)$, derive an estimate for the error $\|u - u_h\|_{H^1(\Omega)}$. Your final estimate should reflect the correct order of convergence with respect to the mesh parameter h. You may use without proof suitable approximation results for the finite element space V_h.

Problem 2. Consider the unit interval $\Omega = (0, 1)$ and the following 1D parabolic problem:

$$\begin{aligned} \partial_t u(x,t) \ - \ \partial_{xx} u(x,t) \ &= \ f(t,x), \qquad && \text{for } x \in \Omega, \ t \in (0,T], \\ u(t,0) = u(t,1) \ &= \ 0 && \text{for } t \in (0,T], \\ u(0,x) \ &= \ u_0(x), && \text{for } x \in \Omega. \end{aligned}$$

Here, $f(t,x)$ and $u_0(x)$ are given, smooth functions.

(a) Derive the variational (in space) formulation of the above problem. What is a suitable function space V?

(b) Discretize the variational formulation in time only (Rothe's method) with the backward Euler scheme.

(c) Let now $\{\psi_j\}_{j=1}^{\infty} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be an orthonormal (in $L_2(\Omega)$) eigenbasis of $-\partial_{xx}$ with homogeneous Dirichlet boundary conditions with corresponding eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$. That is,

$$-\partial_{xx}\psi_j(x) = \lambda_j \psi_j(x),\ x \in (0,1),\ \psi_j(0) = \psi_j(1) = 0,\ \|\psi_j\|_{L_2(0,1)} = 1,\ \int_0^1 \psi_i \psi_j = \delta_{ij}.$$

Now, write the semi discrete solution defined in part (b) as follows:

$$u_k^n = \sum_{j=1}^{\infty} c_j^n\, \psi_j(x),$$

where k is the time-step size, t_n denotes the time point $t_n = k\,n$. Derive the equation following relation for c_j^n:

$$\frac{c_j^{n+1} - c_j^n}{k} \;+\; \lambda_j c_j^{n+1} \;=\; F_j^{n+1},\ j = 0, 1, 2\ldots$$

where $F_j^{n+1} \;:=\; \int_0^1 f(t_{n+1}, x)\psi_j(x)\mathrm{d}x$.

(d) Prove the coefficientwise stability result

$$|c_j^{n+1}| \leq q_j^{n+1}\, |c_j^0| + k \sum_{m=1}^{n+1} q_j^{n+2-m}\, |F_j^m|, \quad \text{where } q_j \;:=\; \frac{1}{1 + k\lambda_j}.$$

Using that $\|u_k^n\|_{L_2(0,1)} = (\sum_{j=1}^{\infty}(c_j^n)^2)^{1/2}$, conclude that if $f = 0$, then

$$\|u_k^n\|_{L_2(0,1)} \leq \|u_0\|_{L_2(0,1)}.$$

Problem 3. Let K be a nondegenerate triangle in $\mathbf{R}^2$. Let a_1, a_2, a_3 be the three vertices of K. Let $a_{ij} = a_{ji}$ denote the midpoint of the segment (a_i, a_j), $i, j \in \{1, 2, 3\}$ and $i \neq j$. Let $\mathcal{P}^2$ be the set of the polynomial functions over K of total degree at most 2. Let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_{12}, \sigma_{23}, \sigma_{31}\}$ be the functionals (or degrees of freedom) on $\mathcal{P}^2$ defined as

$$\sigma_i(p) := p(a_i),\ i \in \{1, 2, 3\} \qquad \sigma_{ij}(p) := p(a_i) + p(a_j) - 2p(a_{ij}),\ i, j = 1, 2, 3,\ i \neq j.$$

(a) Show that Σ is a unisolvent set for $\mathcal{P}^2$ (this means that any $p \in \mathcal{P}^2$ is uniquely determined by the values of the above degrees of freedom applied to p).
(b) Compute the "nodal" basis $\{\psi_j\}_{j=1}^6$ of $\mathcal{P}^2$ which corresponds to $\{\sigma_1, \ldots, \sigma_{31}\}$.

Hint for part (a) and (b): Use barycentric coordinates.

(c) Given $u \in C^0(K)$, define an interpolation operator I_h by

$$(I_h u)(x) = \sum_{j=1}^{3} \sigma_j(u)\psi_j(x) + \sigma_{12}(u)\psi_4(x) + \sigma_{23}(u)\psi_5(x) + \sigma_{23}(u)\psi_6(x).$$

Show the following:
(i) $I_h u = u$ if $u \in \mathcal{P}^2$.
(ii) There is a constant C independent of u and K such that

$$\|I_h u\|_{L_\infty(K)} \leq C\|u\|_{L_\infty(K)}.$$

(iii) Finally deduce that

$$\|u - I_h u\|_{L_\infty(K)} \leq C \inf_{\chi \in \mathcal{P}^2} \|u - \chi\|_{L_\infty(K)}.$$