COMPLEX ANALYSIS QUALIFYING EXAM
JANUARY 2017.

1. Show that $f(z) = 4/z$ is a bijection from $\{z : |z-1| > 1,\ |z-2| < 2\}$ to the strip $\{z : 1 < \mathrm{Re}(z) < 2\}$.

2. Prove the Analytic Convergence Theorem: Let $U \subset \mathbb{C}$ be an open, connected set and $\{f_n\}$ be a sequence of analytic functions on U. If $f_n \to f$ uniformly on every closed disk in U then f is analytic. Moreover, $f_n' \to f'$ pointwise on U and uniformly on every closed disk in U.

3. Fix $R > 0$. Show that there exists an integer $n > 0$ such that for all $m \geq n$ the polynomial $f_m(z) = \sum_{k=0}^{m} \frac{z^k}{k!}$ has no roots w with $|w| < R$.

4. Prove that
$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}$$

5. State and prove the Schwarz lemma.

6. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R < \infty$. Show that there exists a point w with $|w| = R$ such that f can not be analytically continued to any open set which contains w.

7. Can the function $\mathrm{Re}(z^2)$ be approximated uniformly on the unit circle $\{z : |z| = 1\}$ by rational functions having only simple poles? Explain why or why not.

8. Let f be a non-constant, entire function such that $f(1-z) = 1 - f(z)$. Determine the image of f.

9. Let $\mathcal{F}$ be a family of holomorphic functions on the open unit disk Δ. Suppose that $\mathcal{F}' = \{\, f' \mid f \in \mathcal{F}\}$ is a normal family and there exists a point $p \in \Delta$ such that $\{\, f(p) \mid f \in \mathcal{F}\,\}$ is bounded. Is $\mathcal{F}$ a normal family?

10. Let U be a connected, open set and $\{a_n\}$ be a sequence of distinct points in U which do not have a limit point in U. Fix an integer $k \geq 0$. Does there exist an analytic function f on U with prescribed values $f(a_n), \ldots, f^{(k)}(a_n)$ for each n?