Real analysis qualifying exam

August 2012

Each problem is worth ten points. Work each problem on a separate piece of paper.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that the normed vector space $L^1(X, \mu)$ is complete. You may use any results *except* the convergence of function series.

2. Fix two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ with $\mu(X), \nu(Y) > 0$. Let $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$ be measurable. Suppose $f(x) = g(y)$ $(\mu \otimes \nu)$-a.e. Show that there is a constant $a \in \mathbb{C}$ such that $f(x) = a$ μ-a.e. and $g(y) = a$ ν-a.e.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a Borel measurable function. Suppose for every ball B, f is Lebesgue integrable on B and $\int_B f(x)\,dx = 0$. What can you deduce about f? Justify your answer carefully.

4. Let X be a locally compact Hausdorff space. Denote by $C_0(X)$ the space of complex-valued continuous functions on X which vanish at infinity, and by $C_c(X)$ the subset of compactly supported functions. Use an appropriate version of the Stone-Weierstrass theorem to prove that $C_c(X)$ is dense in $C_0(X)$.

5. Give an example of each of the following. Justify your answers.

(a) A nowhere dense subset of $\mathbb{R}$ of positive Lebesgue measure.
(b) A closed, convex subset of a Banach space with multiple points of minimal norm.

6. Let

$$S = \left\{ f \in L^{\infty}(\mathbb{R}) : |f(x)| \leq \frac{1}{1+x^2} \text{ a.e.} \right\}.$$

Which of the following statements are true? Prove your answers.

(a) The closure of S is compact in the norm topology.
(b) S is closed in the norm topology.
(c) The closure of S is compact in the weak-$*$ topology.

7. Let T be a bounded operator on a Hilbert space $\mathcal{H}$. Prove that $\|T^*T\| = \|T\|^2$. State the results you are using.

8.

(a) Let g be an integrable function on $[0,1]$. Does there exist a bounded measurable function f such that $\|f\|_\infty \neq 0$ and $\int_0^1 fg\,dx = \|g\|_1 \|f\|_\infty$? Give a construction or a counterexample.
(b) Let g be a bounded measurable function on $[0,1]$. Does there exist an integrable function f such that $\|f\|_1 \neq 0$ and $\int_0^1 fg\,dx = \|g\|_\infty \|f\|_1$? Give a construction or a counterexample.

9. Let $F : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function, μ the Lebesgue measure, and $f, g \in L^1(\mu)$. Let

$$\widetilde{f}(x) = \int F(xy)f(y)\,d\mu(y), \quad \widetilde{g}(x) = \int F(xy)g(y)\,d\mu(y).$$

Show that $\widetilde{f}$ and $\widetilde{g}$ are bounded continuous functions which satisfy

$$\int f\widetilde{g}\,d\mu = \int \widetilde{f}g\,d\mu.$$

10. Let $\mu, \{\mu_n : n \in \mathbb{N}\}$ be finite Borel measures on $[0,1]$. $\mu_n \to \mu$ *vaguely* if it converges in the weak-$*$ topology on $M[0,1] = (C[0,1])^*$. $\mu_n \to \mu$ *in moments* if for each $k \in \{0\} \cup \mathbb{N}$, $\int_{[0,1]} x^k\,d\mu_n(x) \to \int_{[0,1]} x^k\,d\mu(x)$. Show that $\mu_n \to \mu$ vaguely if and only if $\mu_n \to \mu$ in moments.