Real Analysis Qualifying Examination, January 2010

1. Is it possible to find uncountably many disjoint measurable subsets of $\mathbb{R}$ with strictly positive Lebesgue measure?

2. Let f be a non-negative element of $L_1([0,1])$. Prove that

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{f(x)}dx = m(\{x : f(x) > 0\}).$$

3. (a) Let X be a Banach space with a closed subspace E. If $x \in X$, prove that there exists $\phi \in X^*$ such that $\|\phi\| = 1$, $\phi|_E = 0$, and

$$\phi(x) = \text{dist}\,(x, E).$$

(b) Taking $X = C[-1,1]$ and E to be the subspace of even functions $(f(t) = f(-t))$, consider an odd function $g \in X$ $(g(-t) = -g(t))$. Prove that there exists $\phi \in X^*$, $\|\phi\| = 1$, $\phi|_E = 0$, and

$$\phi(g) = \|g\|_\infty.$$

4. Let m be Lebesgue measure on $[0,1]$. If $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are orthonormal bases for $L_2([0,1],m)$, prove that $\{f_k(x)g_\ell(y)\}_{k,\ell=1}^\infty$ is an orthonormal basis for $L_2([0,1]\times[0,1], m\times m)$.

5. In $C[0,1]$, let

$$A = \text{span}\,\{x^n(1-x) : n \geq 1\}.$$

Prove that A is an algebra whose uniform closure is

$$\{f \in C[0,1] : f(0) = f(1) = 0\}.$$

6. a) State the Riesz representation theorem for the dual of $L_p(\mu)$, where μ is a σ-finite measure on some measurable space (Ω, Σ, μ), and $1 \le p < \infty$.

   b) Prove the following part of the above theorem (you can assume that μ is finite): Let $F \in L_p(\mu)^*$. Then there is a $g \in L_1(\mu)$ so that

$$\int_A g\, d\mu = F(\chi_A),$$

   for all $A \in \Sigma$.

7. Let $1 < p < \infty$ and $f \in L_p[0, \infty)$. Show that

   a)
$$\Big|\int_0^x f(t)\, dt\Big| \le \|f\|_p x^{1-\frac{1}{p}}, \text{for } x > 0.$$

   b)
$$\lim_{x\to\infty} \frac{1}{x^{1-\frac{1}{p}}} \int_0^x f(t)dt = 0.$$

   Hint for part (b): first assume that f has compact support.

8. Let X be a finite dimensional vector space. If $\|\cdot\|$ is a norm on X, prove that $(X, \|\cdot\|)$ is complete. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X, prove that there exist constants $c, k > 0$ so that

$$c\|x\|_1 \le \|x\|_2 \le k\|x\|_1, \quad x \in X.$$

   Hint: note that without loss of generality $\|\cdot\|_1$ can be chosen to be, say, the ℓ_1-norm with respect to some basis.

9. Let P be the vector space of all polynomials with real coefficients. Show there is no norm on P which turns P into a Banach space.
   Hint: You may use the first statement of Problem 8 even if you have been unable to prove it.

10. Let $p \in [1, \infty)$. Show that the unit ball of $L_\infty[0, 1]$ is weakly closed in $L_p[0, 1]$.