Topology/Geometry Qualifying Exam - August 2018

Answer all questions. Write your name and page number in the upper right corner of each page. Start each problem on a new sheet of paper, and use only one side of each sheet.

1. (a) Let $X = \mathbb{N}_+ = \{1, 2, 3, \cdots\}$ and $Y = \{1/n \mid n \in \mathbb{N}_+\}$ be equipped with the subspace topology from $\mathbb{R}$. Here $\mathbb{R}$ is the set of real numbers. Prove that X and Y are homeomorphic.

   (b) Let $A = \mathbb{N} = \{0, 1, 2, 3, \cdots\}$ and $B = \{1/n \mid n \in \mathbb{N}_+\} \cup \{0\}$ be equipped with the subspace topology from $\mathbb{R}$. Prove that A and B are not homeomorphic.

2. Write down explicitly the fundamental groups of $S^2 \times S^1$ and $T^3 = S^1 \times S^1 \times S^1$. Let X be the connected sum of $S^2 \times S^1$ and T^3. Compute the fundamental group of X.

3. Let X be a locally compact Hausdorff space. A continuous function f on X is said to vanish at infinity if the following condition is satisfied: for $\forall \varepsilon > 0$, there exists a compact subset $K \subset X$ such that $|f(x)| < \varepsilon$ for all $x \in X \backslash K$.

   Show that a continuous function f on X extends to a continuous function on X^+ the one-point-compactification of X if and only if there exists a $\lambda \in \mathbb{R}$ such that $f - \lambda$ vanishes at infinity.

4. (a) Prove that S^2 does not admit a continuous tangent vector field that is nonwhere vanishing.

   (b) Construct a continuous tangent vector field of S^3 that is nowhere vanishing.

   (c) Show that there exists a nowhere vanishing a continuous tangent vector field on the 3-dimensional real projective plane $\mathbb{RP}^3$.

5. Let M be the open first quadrant of $\mathbb{R}^2$ and let $F : M \to M$ be the map $F(x, y) = (xy, y/x)$.

   (1) Show that F is a diffeomorphism.

   (2) Compute the push-forward $F_*(X)$, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

6. Let

$$X = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y}$$

be two vector fields on $\mathbb{R}^3$. Let D be the distribution spanned by X and Y.

(1) Find an integral sub-manifold of D passing through the origin.

(2) Compute the Lie bracket $[X, Y]$ of X and Y.

(3) Is the distribution D integrable?

7. Show that the differential 1-form

$$\frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$$

on $\mathbb{R}^2 \setminus \{0\}$ is closed but not exact.

8. (a) Consider the de Rham complex of $\mathbb{R}$:

$$0 \to \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R}) \to 0$$

Prove that $H^0(\mathbb{R}) = \mathbb{R}$ and $H^1(\mathbb{R}) = 0$.

(b) Consider the de Rham complex with compact support of $\mathbb{R}$:

$$0 \to \Omega_c^0(\mathbb{R}) \to \Omega_c^1(\mathbb{R}) \to 0$$

Prove that $H_c^0(\mathbb{R}) = 0$ and $H_c^1(\mathbb{R}) = \mathbb{R}$.