CD Exam, Solutions
Texas A&M High School Math Contest
Nov. 8, 2014

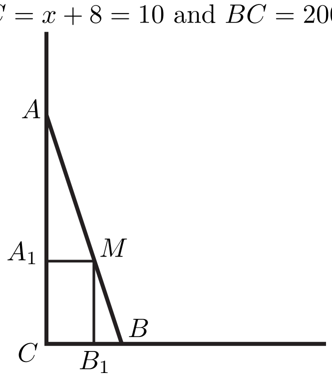
1. If x is the number of coins of each kind, then $273 = x + 5x + 10x + 25x + 50x = 91x$, hence $x = 3$, so we have 15 coins in total.

2. Unpainted cubes must be inside the $8 \times 8 \times 8$ cube, so there are 512 of them, hence there are 1000-512=488 painted cubes.

3. We have $m^2/100$ ounces of acid in the solution. After water is added we get $m + x$ ounces of solution with the same amount of acid, hence $m - 10 = \frac{m^2}{m+x}$, which is equivalent to $m^2 + mx - 10m - 10x = m^2$, hence $(m - 10)x = 10m$, so the answer is $x = \frac{10m}{m-10}$.

4. We have $x^2 - 2xy + y^2 = 1$, and $x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 7$, hence $x^2 + xy + y^2 = 7$. Subtracting $x^2 - 2xy + y^2$ from $x^2 + xy + y^2$, we get $3xy = 6$, hence $xy = 2$. Using $x - y = 1$, we get that $(1 + y)y = 2$, or $y^2 + y - 2 = 0$, which has solutions $y = 1$ and $y = -2$. Hence $(x, y) = (2, 1)$ or $(x, y) = (-1, -2)$

5. Let ABC be the triangle, where AB is the hypotenuse, AC is the leg containing the foot of the perpendicular MA_1 of length 4, and BC is the leg containing the foot of the perpendicular MB_1 of length 8, see the figure. Then $\triangle AA_1M$ is similar to $\triangle MB_1B$, hence $AA_1 : 4 = 8 : BB_1$. Denote $AA_1 = x$. Then $BB_1 = 32/x$, and area of $\triangle ABC$ is $(x + 8)(32/x + 4)/2 = 100$, hence $(x + 8)(32 + 4x) = 200x$, and we get quadratic equation $4x^2 - 136x + 256 = 0$, or $x^2 - 34x + 64 = 0$. Its discriminant is $34^2 - 4 \cdot 64 = 34^2 - 16^2 = (34 - 16)(34 + 16) = 18 \cdot 50 = 36 \cdot 25 = 30^2$, hence $x = \frac{34 \pm 30}{2}$, so x is either 2 cm or 32 cm. In the first case, $AC = x + 8 = 10$ and $BC = 200/AC = 20$. In the second case, $AC = x + 8 = 40$ and $BC = 200/40 = 5$.



6. Multiplying the last two equations, we get $xy + yz + xz = 12$, hence the expression $(a - x)(a - y)(a - z)$ is equal to $a^3 - 6a^2 + 12a - 8 = (a - 2)^3$, hence $x = y = z = 2$ is the only solution.

7. The first terms of the sequence are

$$1, 3, 2, -1, -3, -2, 1, 3, 2, \ldots$$

hence it is periodic with period $(1, 3, 2, -1, -3, -2)$. Note that sum of the entries in the period is equal to 0. The length of the period is 6, the remainder of division of 100 by 6 is 4, hence the sum of the first 100 terms is equal to 1+3+2-1=5.

8. A power a^b, for $a > 0$, is equal to 1 only if either $a = 1$ or $b = 0$ (consider the logarithm $b \log a = 0$). For our equation, in the first case we have $x^2 - x - 1 = 1$, or $x^2 - x - 2 = 0$, which has roots $x = 2$ and $x = -1$. In the second case $x^2 = 1$, which has roots $x = 1$ and $x = -1$.

If $a < 0$ then $a^b = 1$ only if b is an even integer and $a = -1$. If $x^2 - x - 1 = -1$, then $x^2 - x = 0$, which gives values $x = 0$ and $x = 1$. In the first case $x^2 - 1$ is odd.

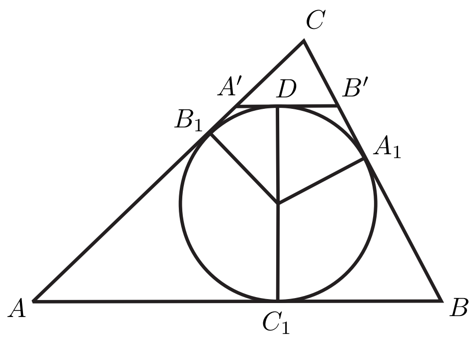
Hence, solutions are $x = 2, 1$, and -1.

9. Let A_1, B_1, C_1, D be the point of tangency of the circle to the lines $BC, AC, AB, A'B'$, respectively. We have $AC_1 = AB_1$, $BC_1 = BA_1$, $A'B_1 = A'D$, $B'D = B'A_1$. It follows that perimeter of $\triangle A'B'C$ is equal to

$20 - 2x$, where $x = AB$. Triangles $\triangle ABC$ and $\triangle A'B'C$ are similar, hence

$$2.4 : x = (20 - 2x) : 20$$

hence $20x - 2x^2 = 48$, or $x^2 - 10x + 24 = 0$, which has solutions $x = 6$ and $x = 4$.



10. Taking logarithms, we get

$$\begin{cases} \ln x \ln 3 = \ln y \ln 4 \\ \ln 4(\ln 4 + \ln x) = \ln 3(\ln 3 + \ln y) \end{cases}$$

We get from the first equation that $\ln y = \frac{\ln x \ln 3}{\ln 4}$. Let us substitute it into the second equation:

$$\ln 4(\ln 4 + \ln x) = \ln 3 \left(\ln 3 + \frac{\ln x \ln 3}{\ln 4} \right)$$

or

$$(\ln 4)^3 + (\ln 4)^2 \ln x = (\ln 3)^2 \ln 4 + (\ln 3)^2 \ln x$$

hence

$$\ln x = \frac{\ln 4((\ln 3)^2 - (\ln 4)^2)}{(\ln 4)^2 - (\ln 3)^2} = -\ln 4,$$

and

$$\ln y = \frac{-\ln 4 \ln 3}{\ln 4} = -\ln 3,$$

hence $x = 1/4$ and $y = 1/3$.

11. For $x \leq 1$, the inequality is equivalent to $1 - x + 2 - x > 3 + x$, or $3 - 2x > 3 + x$, which is equivalent to $x < 0$.

For $1 \leq x \leq 2$, the inequality is equivalent to $x - 1 + 2 - x > 3 + x$, which implies $x < -2$, which is a contradiction.

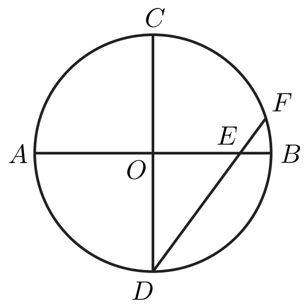
For $x \geq 2$ we have $x - 1 + x - 2 > 3 + x$, which is equivalent to $x > 6$.

Thus, the set of solutions is equal to the union of the intervals $(-\infty, 0)$ and $(6, +\infty)$.

12. Angles $\angle DFB$ and $\angle DAB$ are equal. Similarly, angles $\angle FBA$ and $\angle FDA$ are equal. It follows that triangles $\triangle EFB$ and $\triangle EAD$ are similar, which implies that $AE : ED = FE : EB$, hence $AE \cdot EB = ED \cdot FE$. Let r be the radius of the circle. Then $OE = \sqrt{36 - r^2}$, so $AE = r + \sqrt{36 - r^2}$ and $EB = r - \sqrt{36 - r^2}$. We have therefore

$$r^2 - (36 - r^2) = 12$$

hence $2r^2 = 48$, or $r^2 = 24$. Consequently, area of the circle is $\pi r^2 = 24\pi$.



13. We have $E(100)=0$, so we can ignore it. We can also write numbers with leading zeros: $00, 01, 02, 03, \ldots, 99$, and include 00, since this does not change the sums of even digits. Then each digit appears exactly 20 times. Consequently, $E(1)+E(2)+\cdots+E(100)=20(0+2+4+6+8)=400$.

14. We have

$$x+3-4\sqrt{x-1}=x-1-2\cdot 2\sqrt{x-1}+4=(2-\sqrt{x-1})^2$$

and

$$x+8-6\sqrt{x-1}=x-1-2\cdot 3\sqrt{x-1}+9=(3-\sqrt{x-1})^2$$

It follows that the equation is equivalent to

$$|2-\sqrt{x-1}|+|3-\sqrt{x-1}|=1.$$

If $\sqrt{x-1}\le 2$, then

$$5-2\sqrt{x-1}=1, \quad \text{hence} \quad \sqrt{x-1}=2,$$

and $x=5$. If $2\le\sqrt{x-1}\le 3$, then

$$\sqrt{x-1}-2+3-\sqrt{x-1}=1$$

is always true. This gives the interval $5\le x\le 10$.

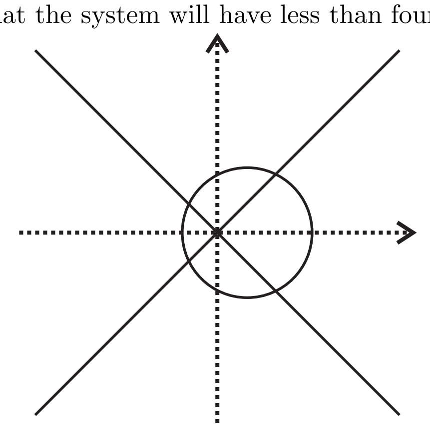
If $3\le\sqrt{x-1}$, then we get

$$2\sqrt{x-1}-5=1, \quad \text{hence} \quad \sqrt{x-1}=3,$$

and $x=10$.

Consequently, the set of solutions is the interval $5\le x\le 10$.

15. Graph of the first equation is union of the lines $x=y$ and $x=-y$. Graph of the second equation is the circle of radius 1 with center in $(a,0)$. In order these graphs to have four intersection points, the intersection point $(0,0)$ of the two lines must be inside the circle. This happens when a is between -1 and 1. It follows that the system will have less than four solutions when either $a\le -1$, or $a\ge 1$.



16. We have $x^8-16=(x^4+4)(x^4-4)$. The second factor is equal to $(x^2-2)(x^2+2)$. For the first factor, we have

$$x^4+4=x^4+4x^2+4-4x^2=(x^2+2)^2-(2x)^2=(x^2+2x+2)(x^2-2x+2).$$

Consequently, $x^8-16=(x^2+2x+2)(x^2-2x+2)(x^2-2)(x^2+2)$.

17. One can show (by induction, or using the formula for the sum of an arithmetic progression) that $1+3+5+\cdots+2n-1=n^2$ and $2+4+6+\cdots+2n=n(n+1)$. Hence the equation is equivalent to $\frac{n^2}{n(n+1)}=\frac{2014}{2015}$, hence $n=2014$.

18. Continue the lines $\overline{DA}$ and $\overline{CB}$ to their intersection point P. Then $\triangle ABP$, $\triangle MNP$, and $\triangle DCP$ are similar. It follows that there exists a coefficient c such that area of the triangles $\triangle ABP$, $\triangle MNP$ and $\triangle DCP$ are equal to cAB^2, cMN^2, and cDC^2, respectively.

Since areas of the trapezoids $ABMN$ and $MNCD$ are equal, area of $\triangle MNP$ is equal to half of the sum of areas of $\triangle ABP$ and $\triangle DCP$. Consequently, $cMN^2=\frac{cAB^2+cDC^2}{2}$, hence $MN^2=\frac{a^2+b^2}{2}$, and $MN=\sqrt{\frac{a^2+b^2}{2}}$.

19. The expression is equal to $\pm(a_1-a_2)\pm(a_2-a_3)\pm(a_3-a_4)\cdots\pm(a_{2013}-a_{2014})\pm(a_{2014}-a_1)$.

After we open the brackets, we will get 2014 numbers with plus sign and 2014 numbers with minus sign. Each of the numbers a_i appears twice. Hence, the maximal possible value is obtained when we make the numbers with plus as big as we can, and numbers with minus as small as we can, i.e., if we get $2\cdot 2014+2\cdot 2013+\cdots+2\cdot 1008-2\cdot 1007-2\cdot 1007-\cdots-2\cdot 2-2\cdot 1=2\cdot\frac{2014\cdot 2015}{2}-4\frac{1007\cdot 1008}{2}=2014\cdot 2015-2\cdot 1007\cdot 1008=2014\cdot(2015-1008)=2014\cdot 1007=2,028,098$.

This sum is realized if we order the numbers into the sequence $2014, 1, 2013, 2, 2012, 3, \ldots, 1008, 1007$.

20. Suppose that $c\geq 0$. Then the equations are $|a+b|+c=19$ and $ab+c=27$. Subtracting them, we get $ab-|a+b|=8$, which is equivalent to $1+ab-|a+b|=9$. This equation is equivalent either to $(a+1)(b+1)=9$ (if $a+b<0$) or to $(a-1)(b-1)=9$ (if $a+b>0$). Each of these equations has 3 solutions:

$$a+1=-1,\quad b+1=-9$$

$$a+1=-3,\quad b+1=-3$$

$$a+1=-9,\quad b+1=-1$$

if $a+b<0$ and

$$a-1=1,\quad b-1=9$$

$$a-1=3,\quad b-1=3$$

$$a-1=9,\quad b-1=1$$

if $a+b>0$. Note that in all these cases $|a+b|$ is equal to 12 or 8, so that c is positive.

If $c<0$, then the equations are $|a+b|+c=19$ and $ab-c=27$. Adding them, we get $ab+|a+b|=46$, which is equivalent to $1+ab+|a+b|=47$. In the same way as above, if $a+b>0$, then we get $(a+1)(b+1)=47$, hence

$$a+1=1,\quad b+1=47$$

or

$$a+1=47,\quad b+1=1.$$

If $a+b<0$, then we have $(a-1)(b-1)=47$, hence

$$a-1=-1,\quad b-1=-47$$

or

$$a-1=-47,\quad b-1=-1.$$

In all cases we get $|a+b|=46$, and $c=-27$ is negative.

Consequently, we get in total 10 solutions.

21. It follows from the equality that $P(0) = 0$. Substituting $x = 1$, we get $-9P(1) = P(0) = 0$, hence $P(1) = 0$. Substituting $x = 2$, we get $-8P(2) = 2P(1) = 0$, hence $P(2) = 0$, etc.. We get in this way that $P(x) = 0$ for $x = 0, 1, 2, \ldots, 9$. It follows that $P(x) = x(x-1)(x-2)\cdots(x-9)Q(x)$ for some polynomial $Q(x)$. Substituting this into the equation, we get

$$x(x-1)(x-2)(x-3)\cdots(x-10)Q(x-1) = (x-10)x(x-1)(x-2)\cdots(x-9)Q(x),$$

hence $Q(x-1) = Q(x)$ for all x, which is possible only if $Q(x)$ is a constant. Hence $P(x) = cx(x-1)(x-2)\cdots(x-9)$, where c is a constant.