DE EXAM

Texas A&M High School Math Contest

November 2014

Directions: If units are involved, include them in your answer.

1. Consider the following system of equations: $\begin{cases} x^4 + y^4 &= 17 \\ x + y &= 3. \end{cases}$ If $x < y$, find $x^{2014} + 2015x - y$.

2. Determine $x^{14} + x^{-14}$, where x satisfies the equation $x^2 + x + 1 = 0$.

3. Find the last digit of S, where $S = 0^2 + 1^2 + 2^2 + \ldots + 99^2$.

4. How many solutions does the system

$$\sin(x + y) = 0$$
$$\sin(x - y) = 0$$

   have, if $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$?

5. Let $p(x) = x(x + 1)(x + 2)(x + 3)$. Find the minimal value of $p(x)$.

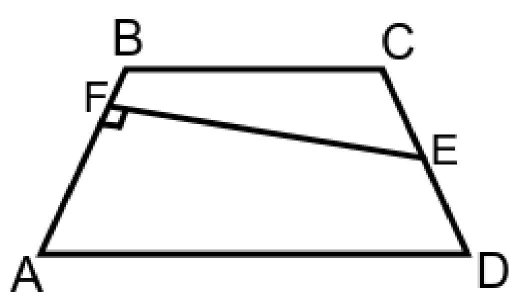
6. A two digit number is equal to the sum of the square of the units digit and the cube of the tens digits. Find this number.

7. Given $x^4 + x^3 + 2x^2 + sx + t$ is a perfect square polynomial (i.e. there is a polynomial $p(x)$ such that $x^4 + x^3 + 2x^2 + sx + t = [p(x)]^2$), find $s^2 + t$.

8. Find $\frac{x^2+y^2}{x^2-y^2} + \frac{x^2-y^2}{x^2+y^2}$ if $\frac{x+y}{x-y} + \frac{x-y}{x+y} = 3$.

9. How many natural numbers are solutions to the equation $2n - 3 = \dfrac{1 - 2n^4}{n^5}$?

10. In the trapezoid $ABCD$ ($AD||BC$), $AB = 5$ and EF is perpendicular to AB, where E is the center of CD (see the picture below). Find the area of the trapezoid, if $EF = 4$.



11. If $\tan\alpha + \tan\beta = 2$ and $\cot\alpha + \cot\beta = 3$, find $\tan(\alpha + \beta)$.

12. Let $0 \le y \le \pi$. Find $x + y$, where x and y are solutions of the following system:

$$\begin{aligned} x^2 + 2014\sin^2 y - 2014 &= 0, \\ \cos x - 2\cos^2 y - 1 &= 0. \end{aligned}$$

13. Find the maximal value of the function $f(x, y) = x + y$ subject to the following conditions:

$$\begin{aligned} &(2\sin x - 1)(2\sqrt{3}\cos y - 3) = 0, \\ &0 \le x \le \tfrac{3\pi}{2}, \quad \pi \le y \le 2\pi. \end{aligned}$$

14. Let x be an integer number such that two of the inequalities

$$2x > 70, \quad x < 100, \quad 4x > 25, \quad x > 5$$

are true, and other two are false. Find x.

15. How many positive integers n, not exceeding 2014, are there such that the sum $1^n + 2^n + 3^n + 4^n$ ends in zero?

16. Let $f(x) = Ax^2 - Ax + 1$, where A is a positive real number. Find the maximal possible value of A such that $|f(x)| \le 1$ for $0 \le x \le 1$.

17. Consider the function $f(x)$ defined on the interval $(0, +\infty)$ with the following properties:

    (a) $f(x) > 0$ for all x;

    (b) $f(1) + f(2) = 10$;

    (c) $f(x + y) = f(x) + f(y) + 2\sqrt{f(x)f(y)}$ for all x, y.

    Find $f(2^{2014})$.

18. Bob and Alice run 10 kilometers. They start at the same place, run 5 kilometers up a hill, and return to the starting point by the same route. Bob has a 10-minute head start and runs at the rate 15km/hour uphill and 20km/hour downhill. Alice runs 16km/hour uphill and 22km/hour downhill. How far from the top of the hill are they when they pass going in opposite directions?

19. A company sells almond butter in cylindrical cans. Marketing research suggests that using wider cans will increase sales. Suppose that the radius of the cans is increased by 25% without altering the volume. By what percent must the height be decreased?

20. Let $\Pi(n)$ and $\Sigma(n)$ denote the product and sum, respectively, of the digits of the integer n. For example, $\Pi(72) = \Pi(27) = 14$ and $\Sigma(72) = \Sigma(27) = 9$. Let N be a two-digit integer such that $N = \Pi(N) + \Sigma(N)$. What is the units digit of N?