SOLUTIONS DE EXAM
Texas A&M High School Math Contest
November 2014

Directions: If units are involved, include them in your answer.

1. Consider the following system of equations:

$$\begin{cases} x^4 + y^4 &= 17 \\ x + y &= 3. \end{cases}$$

If $x < y$, find $x^{2014} + 2015x - y$.

Solution. Let $xy = u$. Then $x^2 + y^2 = 9 - 2xy = 9 - 2u$ and $(9 - 2u)^2 - 2u^2 = x^4 + y^4 = 17$. The last equality implies $u^2 - 18u + 32 = 0$, i.e. $u = 2$ or $u = 16$. So, the given system is equivalent to

$$\begin{cases} xy &= 2 \\ x + y &= 3 \end{cases} \quad \text{or} \quad \begin{cases} xy &= 16 \\ x + y &= 3 \end{cases}$$

The first system has a unique solution satisfying the condition $x < y$. Namely, $(x, y) = (1, 2)$. The second one does not have a solution.

Thus, $x^{2014} + 2015x - y = 1^{2014} + 2015 \cdot 1 - 2 = 2014$.

Answer: 2014

2. Determine $x^{14} + x^{-14}$, where x satisfies the equation $x^2 + x + 1 = 0$.

Solution. First note that $x \neq 0$. Dividing both sides of the given equation by x, we get $x+1+x^{-1} = 0$, or $x+x^{-1} = -1$. Multiplying both sides of the given equation by x, we get $x^3+x^2+x = 0$, or $x^3 = -(x^2+x) = -(-1) = 1$ (by the given equation). Hence, $x^{14}+x^{-14} = (x^3)^4x^2+(x^3)^{-2}x^{-2} = x^2 + x^{-2} = (x + x^{-1})^2 - 2 = 1 - 2 = -1$.

Answer: -1

3. Find the last digit of S, where $S = 0^2 + 1^2 + 2^2 + \ldots + 99^2$.

Solution. Let us denote by $\phi(A)$ the last digit of a number A. If

$$\begin{array}{ccccccccccc} 0^2 & + & 1^2 & + & 2^2 & + & \ldots & + & 9^2 & = & S_0, \\ 10^2 & + & 11^2 & + & 12^2 & + & \ldots & + & 19^2 & = & S_1, \\ & & & & & & \ldots & & & & \\ 90^2 & + & 91^2 & + & 92^2 & + & \ldots & + & 99^2 & = & S_9, \end{array}$$

then $\phi(S_0) = \phi(S_1) = \ldots = \phi(S_9)$. Since
$S = S_0 + S_1 + \ldots + S_9$, we get

$$\phi(S) = \phi(\phi(S_0) + \phi(S_1) + \ldots + \phi(S_9)) = \phi(10 \cdot \phi(S_0)) = 0.$$

Answer: 0

4. How many solutions does the system

$$\sin(x+y) = 0$$
$$\sin(x-y) = 0$$

have, if $0 \le x \le \pi$ and $0 \le y \le \pi$?

Solution. The first equation implies $x + y = n\pi$ for some integer n. The second equation yields $x - y = m\pi$ for some integer m. Since we are looking for a solution such that both x and y are from 0 to π, we conclude that

$$0 \le m + n \le 2,$$
$$0 \le n - m \le 2.$$

Note that the only values n can take are $0, 1, 2$. If $n = 0$, then $m = 0$. If $n = 1$, then $m = -1, 0, 1$. If $n = 2$, then $m = 0$. So, there are 5 solutions in total: $(x, y) = (0, 0), (0, \pi), (\pi/2, \pi/2), (\pi, 0), (\pi, \pi)$.

Answer: 5

5. Let $p(x) = x(x + 1)(x + 2)(x + 3)$. Find the minimal value of $p(x)$.

Solution. Note that $p(x) = x(x+1)(x+2)(x+3) = x(x+3)\cdot(x+1)(x+2) = (x^2+3x)(x^2+3x+2)$. Let $z = x^2 + 3x$. Then $p(x) = z(z + 2) = z^2 + 2 = (z + 1)^2 - 1$. It follows that the minimal value of $p(x)$ is -1 and it is attained when $z = -1$. Since the equation $x^2 + 3x = -1$ has real solutions (the discriminant of this quadratic equation is positive), we conclude that there is x such that $p(x) = -1$.

Answer: -1

6. A two digit number is equal to the sum of the square of the units digit and the cube of the tens digits. Find this number.

Solution. Let a be the units digit and b be the tens digit of the given number. Then

$$10b + a = a^2 + b^3, \tag{1}$$

or $b(10 - b^2) = a(a - 1)$. Note that the left hand side of the last equality is always positive and even. Therefore b is also even and $10 - b^2 > 0$. These yields $b \le 3$, and then $b = 2$. If $b = 2$ then (1) implies $a^2 - a - 12 = 0$. The only positive root of this equation is $a = 4$.

Answer: 24

7. Given $x^4 + x^3 + 2x^2 + sx + t$ is a perfect square polynomial (i.e. there is a polynomial $p(x)$ such that $x^4 + x^3 + 2x^2 + sx + t = [p(x)]^2$), find $s^2 + t$.

Solution. First note that that the polynomial $p(x)$ must be quadratic. Thus, let $p(x) = ax^2+bx+c$. Then $x^4 + x^3 + 2x^2 + sx + t = (ax^2 + bx + c)^2$. For the two sides of the last equality to be equal, the coefficients of the two polynomials must be equal. So, equating the coefficients, we get $a = 1, 2b = 1, b^2 + 2c = 2, 2bc = s, c^2 = t$. Solving this system, we obtain $a = 1, b = 1/2, c = s = 7/8, t = 49/64$. So, $s^2 + t = \frac{49}{32}$

Answer: $\frac{49}{32}$

8. Find

$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}$$

if

$$\frac{x+y}{x-y}+\frac{x-y}{x+y}=3.$$

Solution. We have

$$3=\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{2(x^2+y^2)}{x^2-y^2}.$$

Therefore,

$$\frac{x^2+y^2}{x^2-y^2}=\frac{3}{2}$$

and

$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}=\frac{3}{2}+\frac{2}{3}=\frac{13}{6}.$$

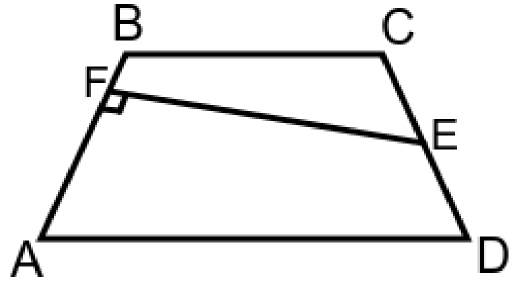
Answer: 13/6

9. How many natural numbers are solutions to the equation $2n-3=\dfrac{1-2n^4}{n^5}$?

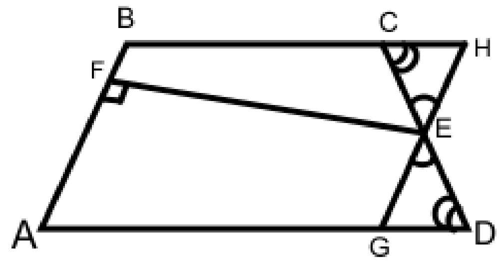
Solution. Rewrite the given equation in the form $2n^6-3n^5+2n^4=1$, or $n(2n^5-3n^4+2n^3)=1$. Since the left hand side of the last equation is divisible by n, then the right hand side must be divisible by n. Hence, $n=1$ is the only natural solution of the given equation.

Answer: 1

10. In the trapezoid $ABCD$ ($AD||BC$), $AB=5$ and EF is perpendicular to AB, where E is the center of CD (see the picture below). Find the area of the trapezoid, if $EF=4$.



Solution. Let GH be parallel to AB as it is shown on the picture below.



Then ABHG is a parallelogram. Moreover, the triangles GED and CHE are congruent (two angles and the included side are the same). Thus,

$$Area(ABCD) = Area(ABHG) = 2Area(ABE) = 2(\frac{1}{2}AB \cdot EF) = 5 \cdot 4 = 20.$$

Answer: 20

11. If $\tan\alpha + \tan\beta = 2$ and $\cot\alpha + \cot\beta = 3$, find $\tan(\alpha+\beta)$.

Solution. $\dfrac{2}{\tan\alpha\tan\beta} = \dfrac{\tan\alpha+\tan\beta}{\tan\alpha\tan\beta} = \cot\alpha + \cot\beta = 3$. Hence, $\tan\alpha\tan\beta = \frac{2}{3}$ and

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{2}{1-\frac{2}{3}} = 6.$$

Answer: 6

12. Let $0 \le y \le \pi$. Find $x+y$, where x and y are solutions of the following system:

$$\begin{aligned} x^2 + 2014\sin^2 y - 2014 &= 0, \\ \cos x - 2\cos^2 y - 1 &= 0. \end{aligned}$$

Solution. The second equation implies $2\cos^2 y = \cos x - 1 \le 0$ and then $\cos^2 y = 0$, i.e. $y = \pi/2$. Also $\sin^2 y = 1$ and then the first equation yields $x = 0$. So, $x + y = \pi/2$.

Answer: $\pi/2$

13. Find the maximal value of the function $f(x, y) = x + y$ subject to the following conditions:

$$(2\sin x - 1)(2\sqrt{3}\cos y - 3) = 0,$$

$$0 \le x \le \frac{3\pi}{2}, \quad \pi \le y \le 2\pi.$$

Solution. The given equation implies that $\sin x = 1/2$ or $\cos y = \sqrt{3}/2$. In each of these cases, the sum $x + y$ is maximal if each of the terms is maximal. In the first case this means that $x = 5\pi/6$, $y = 2\pi$, and in the second case this means that $x = 3\pi/2$, $y = 11\pi/6$. Since $f(5\pi/6, 2\pi) = 17\pi/6$ and $f(3\pi/2, 11\pi/6) = 10\pi/3$, we conclude that the maximal value of f under the above conditions is $10\pi/3$.

Answer: $10\pi/3$

14. Let x be an integer number such that two of the inequalities

$$2x > 70, \quad x < 100, \quad 4x > 25, \quad x > 5$$

are true, and other two are false. Find x.

Solution. Since x is integer, we can rewrite the given inequalities:

$$\textbf{(a)} \quad x > 35, \qquad \textbf{(b)} \quad x < 100, \qquad \textbf{(c)} \quad x > 6, \qquad \textbf{(d)} \quad x > 5.$$

If **(a)** is false, then the remaining three inequalities are true, which is a contradiction So **(a)** must be false, which then forces **(b)** to be true. This implies exactly one of **(c)** and **(d)** is true. This forces x to satisfy $5 < x \leq 6$. Hence x must equal 6.

Answer: 6

15. How many positive integers n, not exceeding 2014, are there such that the sum $1^n + 2^n + 3^n + 4^n$ ends in zero?

Solution. It is sufficient to consider the following three cases:

Case 1: n is odd. Then 1^n+4^n is also odd and divisible by $1+4=5$. So, it ends in 5. Similarly, $2^n + 3^n$ ends in 5. Therefore, the given sum ends in zero.

Case 2: $n = 4k + 2$ for some non negative integer k. Then $1^n + 2^n$ is odd and divisible by $1^2 + 2^2 = 5$. So, it ends in 5. Similarly, $3^n + 4^n$ is odd and divisible by $3^2 + 4^2 = 25$, i.e. ends in 5. Therefore, the given sum ends in zero again.

Case 3: $n = 4k$ for some positive integer k. Since the last digit of 2^4 and 4^4 is 6, the last digit of $2^n = (2^4)^k$ and $4^n = (4^4)^k$ is also 6. Similarly, the last digit of 3^n is 1. So, the given sum ends by 4 in this case.

There are 503 positive integers divisible by 4 and not exceeding 2014 (note that $503 \cdot 4 = 2012$). So, we obtain $2014 - 503 = 1511$ numbers such that the given sum ends in zero.

Answer: 1511

16. Let $f(x) = Ax^2 - Ax + 1$, where A is a positive real number. Find the maximal possible value of A such that $|f(x)| \leq 1$ for $0 \leq x \leq 1$.

Solution. The graph of f is a parabola. Since $f(0) = f(1) = 1$, this parabola is symmetric about $x = 0.5$. In addition, $|f(x)| \leq 1$ implies that the branches of the parabola are directed upward. The minimal value of $f(x)$ is $f(0.5) = 1 - A/4$. And the maximal possible value of A satisfies the condition $f(0.5) = 1 - A/4 = -1$, i.e. when $A = 8$.

Answer: 8

17. Consider the function $f(x)$ defined on the interval $(0, +\infty)$ with the following properties:

(a) $f(x) > 0$ for all x;

(b) $f(1) + f(2) = 10$;

(c) $f(x + y) = f(x) + f(y) + 2\sqrt{f(x)f(y)}$ for all x, y.

Find $f(2^{2014})$.

Solution. The last property implies that $f(2x) = f(x+x) = 4f(x)$. In particular, $f(2) = 4f(1)$ and combining this with (b) we obtain that $5f(1) = 10$, i.e. $f(1) = 2$. We have $f(x+y) = (\sqrt{f(x)} + \sqrt{f(y)})^2$. This identity implies that for any positive integer n, $f(nx) = n^2 f(x)$. Thus, $f(2^{2014}) = 2^{2028} f(1) = 2^{2049}$.

Answer: 2^{4029}

18. Bob and Alice run 10 kilometers. They start at the same place, run 5 kilometers up a hill, and return to the starting point by the same route. Bob has a 10-minute head start and runs at the rate 15km/hour uphill and 20km/hour downhill. Alice runs 16km/hour uphill and 22km/hour downhill. How far from the top of the hill are they when they pass going in opposite directions?

Solution. Let t be the time (in hrs) since the start of the race and x be the distance from the top of the hill to the point where they meet. Since Alice starts 1/6 hr later, the distance she has traveled uphill is $16(t - 1/6) = 5 - x$. On Bob's trip downhill (1/3 hr after starting), he has traveled the distance x in $t - 1/3$ hrs, so $20(t - 1/3) = x$. Solving these equations yields $t = \frac{43}{108}$ hrs and $x = \frac{35}{27}$ km.

Answer: 35/27 km

19. A company sells almond butter in cylindrical cans. Marketing research suggests that using wider cans will increase sales. Suppose that the radius of the cans is increased by 25% without altering the volume. By what percent must the height be decreased?

Solution. Let $r$and h be the radius and height of the original can and let R and H be the radius and the height of the new can. Then $R = 1.25r = \frac{5}{4}r$. The volume of the can is

$$\pi r^2 h = \pi R^2 H = \pi \left(\frac{5}{4}r\right)^2 H,$$

and

$$H = \frac{\pi r^2}{\pi(\frac{5}{4}r)^2} h = \frac{16}{25} h.$$

Thus the height has been decreased by $(1 - 16/25) = 9/25 = 36/100 = 36\%$.

Answer: 36%

20. Let $\Pi(n)$ and $\Sigma(n)$ denote the product and sum, respectively, of the digits of the integer n. For example, $\Pi(72) = \Pi(27) = 14$ and $\Sigma(72) = \Sigma(27) = 9$. Let N be a two-digit integer such that $N = \Pi(N) + \Sigma(N)$. What is the units digit of N?

Solution. Write N in the form $10a + b$, where a is one of the numbers $1, 2, \cdots, 9$ and b is one of the numbers $0, 1, 2, \cdots, 9$. Then we have:

$$10a + b = N = \Pi(N) + \Sigma(N) = ab + (a + b).$$

So, $9a = ab$ and since $a \neq 0$, we conclude that $b = 9$.

Answer: 9