Solutions to EF Exam

Texas A&M High School Math Contest
08 November, 2014

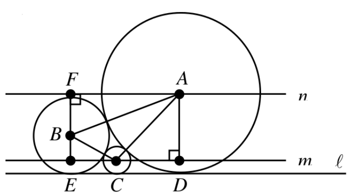
1. Multiplying by $x-4$ yields $x^2-5=2x^2+x-36+11$, or $x^2+x-20=0$. The solutions to this equation are $x=-5$ and $x=4$; however, $x=4$ is not in the domain of the expressions on either side of the equation. Therefore, the only solution (hence the sum of all solutions) is $\mathbf{-5}$.

2. $x(x+y+1)=14$ and $y(x+y+1)=28$, so $y=2x$. Substituting into the first equation yields $3x^2+x-14=0$, or $(3x+7)(x-2)=0$. Therefore, $x=-\frac{7}{3}$ and $y=-\frac{14}{3}$. $\left(-\mathbf{\frac{7}{3}},-\mathbf{\frac{14}{3}}\right)$.

3. Multiply the second equation by 2 and add to the first equation to yield $\frac{5}{\sqrt{x}}=\frac{10}{3}$, or $\sqrt{x}=\frac{3}{2}$. Substitute into either equation to yield $\sqrt{y}=\frac{3}{2}$. So $\sqrt{x}+\sqrt{y}=\mathbf{3}$.

4. Draw $\overline{OA}$ and $\overline{OM}$, forming right triangles WAO and OMN which are similar to $\triangle WIN$ and therefore to each other. $\triangle WAO$ is a 9-12-15 right triangle, which makes $\triangle OMN$ a 12-16-20 right triangle, so $UN=\mathbf{8}$.

5. Draw lines connecting the centers and draw lines m and n parallel to ℓ through C and A respectively as shown below.



Let x be the radius of circle C. Then $CD=\sqrt{(18+x)^2-(18-x)^2}=6\sqrt{2x}$, and $CE=\sqrt{(8+x)^2-(8-x)^2}=4\sqrt{2x}$, so $DE=10\sqrt{2x}$. But $\overline{DE}\cong\overline{AF}$ and $AF=\sqrt{(18+8)^2+(18-8)^2}=24$, so $10\sqrt{2x}=24$. Solving for x yields a radius of $\mathbf{\frac{72}{25}}$.

6. Let x, y, and z be the number of small, medium, and large trucks respectively. The augmented matrix for this system of equations is $\left[\begin{array}{ccc|c}1&1&1&25\\400&800&1600&32000\end{array}\right]$, which row-reduces to $\left[\begin{array}{ccc|c}1&0&-2&-30\\0&1&3&55\end{array}\right]$, meaning $x-2z=-30$ and $y+3z=55$. Since x, y, and z must be nonnegative integers, we have $15\le z\le 18$. **18 large trucks**

7. Let $x=\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}$. Then $x^2=(2+\sqrt{3})+2\sqrt{4-3}+(2-\sqrt{3})=6$, so $x=\sqrt{6}$. Therefore, $n=\mathbf{6}$.

8. Moving all terms to one side and factoring yields $(\sin^2 x-1)(\sec x+2)=0$, so $\sin x=\pm1$, meaning $x=\frac{\pi}{2}$ or $x=\frac{3\pi}{2}$, or $\sec x=-2$, meaning $x=\frac{2\pi}{3}$ or $x=\frac{4\pi}{3}$. However, $\sec x$ is undefined at $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$, so the product of all solutions is $\frac{2\pi}{3}\cdot\frac{4\pi}{3}=\mathbf{\frac{8\pi^2}{9}}$.

9. Let $n=20^m$, where m is any real number. Then $(20^m)^{\log_{20}14}=20^{m\log_{20}14}=14^m=14^2$, so $m=2$ and $n=20^2=\mathbf{400}$.

10. Place \$1 in the first bag, \$2 in the second bag, \$4 in the third bag, and so on, doubling each time, until you place \$256 in the ninth bag. The bags hold a total of \$511, so the last bag contains the remaining **\$489**.

11. Note that the desired result is obtained by expanding the product

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)$$

Each factor above is the sum of a positive geometric series, so the resulting sum is $\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right) = (2)\left(\frac{3}{2}\right) = \mathbf{3}$.

12. The line is tangent to f at $(8, -4)$, so the limit is indeterminate. Either use L'Hospital's Rule or multiply numerator and denominator by $x^{2/3}+2x^{1/3}+4$ to obtain $\lim\limits_{x\to 8}\left(\frac{f(x)-(-4)}{x-8}\right)(x^{2/3}+2x^{1/3}+4) = f'(8)\cdot 12$ From the equation of the tangent line, $f'(8) = -3$, so the limit is $\mathbf{-36}$.

13. $f(x) = \frac{x^{2014}-1}{x-1} + \frac{1}{x-1}$. The first rational expression (call it $f_1(x)$) reduces to a 2013th degree polynomial, so $f_1^{(2014)}(x) = 0$. Using induction, it can be shown that $\frac{d^n}{dx^n}\left(\frac{1}{x-1}\right) = \frac{(-1)^n n!}{(x-1)^{n+1}}$, so $f^{(2014)}(x) = \frac{\mathbf{2014!}}{\mathbf{(x-1)^{2015}}}$.

14. Let (a, b) be the point where ℓ is tangent to the curve. An equation of the line is $y - b = -\frac{b^{1/3}}{a^{1/3}}(x-a)$, or $y = -\frac{b^{1/3}}{a^{1/3}}x + b^{1/3}(a^{2/3}+b^{2/3}) = -\frac{b^{1/3}}{a^{1/3}}x + 4b^{1/3}$. The x-intercept is $(4a^{1/3}, 0)$ and the y-intercept is $(0, 4b^{1/3})$, so the length of the segment of ℓ is $4\sqrt{a^{2/3}+b^{2/3}} = 4\sqrt{4} = \mathbf{8}$ (independent of a and b).

15. $\frac{(x^2+2)(y^2+2)(z^2+2)}{xyz} = \left(x+\frac{2}{x}\right)\left(y+\frac{2}{y}\right)\left(z+\frac{2}{z}\right)$, so the minimum value occurs when x, y, and z are all equal to the value of t which minimizes $f(t) = t + \frac{2}{t}$, $t > 0$. $f'(t) = 1 - \frac{2}{t^2}$, so f has a positive critical value only at $t = \sqrt{2}$, and since $f''(\sqrt{2}) = \frac{2}{2^{3/2}} > 0$, f is minimized at $t = \sqrt{2}$. Therefore, the minimum value of the expression is $\left(\sqrt{2}+\frac{2}{\sqrt{2}}\right)^3 = \mathbf{16\sqrt{2}}$

16. Let (a, a^2) be the coordinates of A, (b, b^2) be the coordinates of B, and (c, c^2) be the coordinates of C. Also assume, without loss of generality, that $a < b$ as shown in the figure. The area of Φ is the area under $\overline{AB}$ minus the area under the parabola, or $\frac{1}{2}(a^2+b^2)(b-a) - \int_a^b x^2\,dx = \frac{1}{2}(a^2+b^2)(b-a) - \frac{1}{3}(b^3-a^3) = \frac{1}{6}(b-a)(3a^2+3b^2-2(b^2+ab+a^2)) = \frac{1}{6}(b-a)(b^2-2ab+a^2) = \frac{1}{6}(b-a)^3$. Using trapezoids drawn to the x-axis, the area of $\triangle ABC$ is $\frac{1}{2}(a^2+b^2)(b-a) - \frac{1}{2}(a^2+c^2)(c-a) - \frac{1}{2}(c^2+b^2)(b-c)$. But the slope of the tangent line is $2c = \frac{b^2-a^2}{b-a} = b+a$, so c is the average of a and b and $c - a = b - c = \frac{1}{2}(b-a)$, so the area is $\frac{1}{4}(b-a)\left(2a^2+2b^2-a^2-c^2-c^2-b^2\right) =$

$$\frac{1}{4}(b-a)(a^2+b^2-2c^2) = \frac{1}{4}(b-a)\left(a^2+b^2-\frac{b^2+2ab+a^2}{2}\right) = \frac{1}{8}(b-a)(b^2-2ab+a^2) = \frac{1}{8}(b-a)^3.$$

Therefore, the ratio of the areas is $\dfrac{\frac{1}{6}(b-a)^3}{\frac{1}{8}(b-a)^3} = \mathbf{\dfrac{4}{3}}$.

17. Let P' be the point $(-2, 0)$. Then $|PR - QR| = |P'R - QR| \leq P'Q$ by the Triangle Inequality, with the maximum value occuring at equality. This occurs when P', Q, and R are collinear. Using vectors or by finding the y-intercept of $\overleftrightarrow{P'Q}$, we find the coordinates of R are $(0, 8)$, so $y = \mathbf{8}$.

18. A line $y = mx + b$ intersects the curve in exactly four points if and only if $f(x) = x^4 + 3x^3 + cx^2 + (2-m)x + (4-b)$ has exactly four zeros, which means the derivative $f'(x) = 4x^3 + 9x^2 + 2cx + (2-m)$ has exactly three zeros and $f''(x) = 12x^2 + 18x + 2c$ has exactly two zeros. This will be true if and only if $18^2 - 4(12)(2c) > 0$, or $c < \dfrac{18^2}{96} = \dfrac{27}{8}$. Conversely, if $c < \dfrac{27}{8}$, f'' has exactly two zeros, so f' has exactly two extrema, so we can choose m so that f' has a zero between the extrema, meaning f' has exactly three zeros. Similarly, we can choose b so that f has exactly four zeros. Therefore, the value of N we seek is $\mathbf{\dfrac{27}{8}}$.

19. Let $y = \sin x + \cos x$. Then $y^2 = \sin^2 x + \cos^2 x + 2\sin x\cos x = 1 + 2\sin x\cos x$, or $\sin x\cos x = \dfrac{y^2-1}{2}$. Therefore, $\tan x + \cot x = \dfrac{1}{\sin x\cos x} = \dfrac{2}{y^2-1}$, and $\sec x + \csc x = \dfrac{\sin x + \cos x}{\sin x\cos x} = \dfrac{2y}{y^2-1}$. Therefore, the given problem is equivalent to minimizing $f(y) = \left|y + \dfrac{2}{y^2-1} + \dfrac{2y}{y^2-1}\right| = \left|y + \dfrac{2}{y-1}\right|$ for all $y \neq \pm 1$ (it can easily be shown that $y = \sin x + \cos x = \pm 1$ if and only if $x = \dfrac{n\pi}{2}$, $n \in Z$). Since $y + \dfrac{2}{y-1} \neq 0$, f is differentiable everywhere on its domain. $f'(y) = \dfrac{y + \frac{2}{y-1}}{\left|y + \frac{2}{y-1}\right|}\left(1 - \dfrac{2}{(y-1)^2}\right) = 0$ when $y = 1 \pm \sqrt{2}$. Since $f(1+\sqrt{2}) = 1 + 2\sqrt{2}$ and $f(1-\sqrt{2}) = |1 - 2\sqrt{2}| = 2\sqrt{2} - 1$, we find the minimum of f is $\mathbf{2\sqrt{2} - 1}$.