AB ExamSolutions
Texas A&M High School Math Contest
November 8, 2014

1. What is the largest power of 2 that divides $2^{2013} + 10^{2013}$?

   ANSWER: 2^{2014}

   Solution: $2^{2013} + 10^{2013} = 2^{2013}(1 + 5^{2013})$. Since $1 + 5^{2013}$ is even, it is divisible by 2. But it is not divisible by 4, so the highest power of 2 is $2^{2013}2 = 2^{2014}$. To see that 5^{2013} is not divisible by 4, note that $5^{2013} = (1 + 4)^{2013}$ is a polynomial in 4 with constant term 1.

2. It is the year 2014, and Gracie just happens to have 2014 pennies. She uses these pennies to buy Q items, each item costing P pennies. She has 14 pennies left over. How many possibilities are there for the price P?

   ANSWER: 20

   Solution: We have $2014 = PQ + 14$ or $PQ = 2014 - 14 = 2000 = 2^4 \cdot 5^3$. There are $5 \cdot 4 = 20$ different factors of 2000. Hence 20 different possible prices.

3. The two roots of the quadratic equation $x^2 - 85x + C = 0$ are prime integers. What is the value of C?

   ANSWER: 166

   Solution: The sum of the two primes is 85 which is an odd integer. So one of the primes must be even, i.e. 2, and the other is $85 - 2 = 83$. this means $C = 2(83) = 166$.

4. Which of the three numbers 2^{100}, 3^{75}, 5^{50} is the largest?

   ANSWER: 3^{75}

   Solution: $2^{100} = 16^{25}$
   $3^{75} = 27^{25}$
   $5^{50} = 25^{25}$
   and 3^{75} is the largest.

5. How many pairs (x, y) of non-negative integers satisfy $x^4 - y^4 = 16$?

ANSWER: One

Solution: Let (x, y) be a pair of integers satisfying $x^4 - y^4 = 16$. Then $(x^2 - y^2)(x^2 + y^2) = 16$. We have the following possibilities and corresponding integer solutions:

$$\begin{aligned} x^2 + y^2 &= 1 & \quad & (\pm 1, 0),\ (0, \pm 1) \\ x^2 + y^2 &= 2 & & (\pm 1, \pm 1) \\ x^2 + y^2 &= 4 & & (\pm 2, 0),\ (0, \pm 2) \\ x^2 + y^2 &= 8 & & (\pm 2, \pm 2) \\ x^2 + y^2 &= 16 & & (\pm 4, 0),\ (0, \pm 4). \end{aligned}$$

Only $(\pm 2, 0)$ satisfies $x^4 - y^4 = 16$.

6. The dimensions of a cylinder are changed as follows: the height is increased by $\frac{1}{9}$th of the old height, and the radius is decreased by $\frac{1}{10}$th of the old radius. What is the ratio of the original cylinder's volume to that of the new cylinder?

ANSWER: $\frac{10}{9}$

Solution: Let the volume of the original cylnder be $V_0 = \pi r_0^2 h_0$. The volume of the new cylinder is $V_n = \pi(h_0 + \frac{1}{9}h_0)(r_0 - \frac{1}{10}r_0)^2 = \pi \frac{10}{9} h_0 \left(\frac{9}{10} r_0\right)^2 = \frac{9}{10}\pi r^2 h$. So $\frac{V_0}{V_n} = \frac{10}{9}$.

7. What is the last (units) digit of 2^{2014}?

ANSWER: 4

Solution: $2^{2014} = (2^2)^{1007} = 4^{1007}$. Since $4^2 = 16$ and $6^2 = 36$ every even power of 4 ends in 6 and $4 \cdot 6 = 24$, so 2^{2014} ends in 4.

8. How many polynomials are there of the form $x^3 - 8x^2 + cx + d$ such that c and d are real numbers and the three roots of the polynomial are distinct positive integers?

ANSWER: 2

Solution: $x^3 - 8x^2 + cx + d = (x - \alpha)(x - \beta)(x - \delta)$ so $\alpha + \beta + \delta = 8$ with α, β, δ positive integers and all different: The possiblities are
$1 + 2 + 5 = 8$
$1 + 3 + 4 = 8.$
So 2.

9. From a group of men and women, 15 women leave. There are then left two men for each woman. From this reduced group 45 men leave. There are then 5 women for each man. How many women were in the original group?

   ANSWER: 40 (women)

   Solution: Let W be the number of women and M the number of men.
   $W - 15 = {}^{1}\!/_{2}M$
   $M - 45 = {}^{1}\!/_{5}(W - 15)$
   $2W - 30 = M$

   $5M - 225 = W - 15$ or $5M = W + 210$
   $10W - 150 = 5M = W + 210$

   $9W = 360$
   $W = 40$

10. Hasse has a 20 gram ring that is 60% gold and 40% silver. He wants to melt it down and add enough gold to make it 80% gold. How many grams of gold should he add?

    ANSWER: 20 (grams)

    Solution: The amount of gold is $(.6)20 = 12$ grams and there are 8 grams of silver. Let x be the amount of gold added for an 80% combination. There will still be 8 grams of silver. $A = 12 + x + 8$ with 20% silver. So

$$\begin{aligned} \tfrac{2}{10}A = \tfrac{2}{10}(20 + x) =& 8 \\ 4 + \tfrac{1}{5}x =& 8 \\ \tfrac{1}{5}x =& 4 \\ x =& 20. \end{aligned}$$

11. On a test the passing students had an average of 83 while the failing students had an average of 55. If the overall class average was 76, what percent of the class passed?

    ANSWER: 75%

    Solution: Let p be the number who passed and f the number who failed. We have $83p$ and $55f$ equal to the points accumulated by those who passed and failed respectively. Thus

$$76 = \frac{83p + 55f}{p + f} = \frac{28p + 55(p + f)}{p + f} = 28\frac{p}{p + f} + 55\,.$$

    Thus, $\dfrac{p}{p + f} = \dfrac{76 - 55}{28} = \dfrac{21}{28} = \dfrac{3}{4}$ or 75%.

12. Find all solutions to $|5x-2|+|5x+1|=3$.

ANSWER: $x \in \left[-\frac{1}{5}, \frac{2}{5}\right]$ or $-\frac{1}{5} \le x \le \frac{2}{5}$.

Solution: Assume $x \ge \frac{2}{5}$, then

$$\begin{aligned} 5x-2+5x+1 &= 3 \\ 10x &= 4 \\ x &= \tfrac{2}{5}. \end{aligned}$$

Assume $-\frac{1}{5} \le x \le \frac{2}{5}$, then

$$\begin{aligned} -5x+2+5x+1 &= 3 \\ 3 &= 3 \end{aligned}$$

and all x satisfy the equation. Assume $x \le -\frac{1}{5}$, then

$$\begin{aligned} -5x+2-5x-1 &= 3 \\ -10x &= 2 \\ x &= -\tfrac{1}{5}. \end{aligned}$$

13. If x^2-2x-3 is a factor of x^4+px^2+q, what is the value of p?

ANSWER: -10

Solution: Since $x^2-2x-3=(x-3)(x+1)$, its roots are 3 and -1. So they must be roots of x^4+px^2+q as well. Hence

$$\begin{aligned} 3^4+9p+q &= 0 \\ 1+p+q &= 0. \end{aligned}$$

Subtracting the 2nd equation from the first gives

$$80+8p=0$$

and $p=-10$.

14. The parabola $y=ax^2+bx+1$ has a maximum at $(2,2)$. What is the value of b?

ANSWER: 1

Solution: Since the parabola has a maximum, it opens downward and is symmetric about the line $x=2$. It contains the points $(2,2)$ and $(0,1)$. By symmetry, it also contains the point $(4,1)$. Substituting $(2,2)$ and $(4,1)$ gives

$$\begin{aligned} 2 &= 4a+2b+1 \\ 1 &= 16a+4b+1. \end{aligned}$$

Solving for b gives $b=1$.

15. For all real numbers x and y that satisfy $(x+5)^2 + (y-12)^2 = 14^2$, find the minimum value of $x^2 + y^2$.

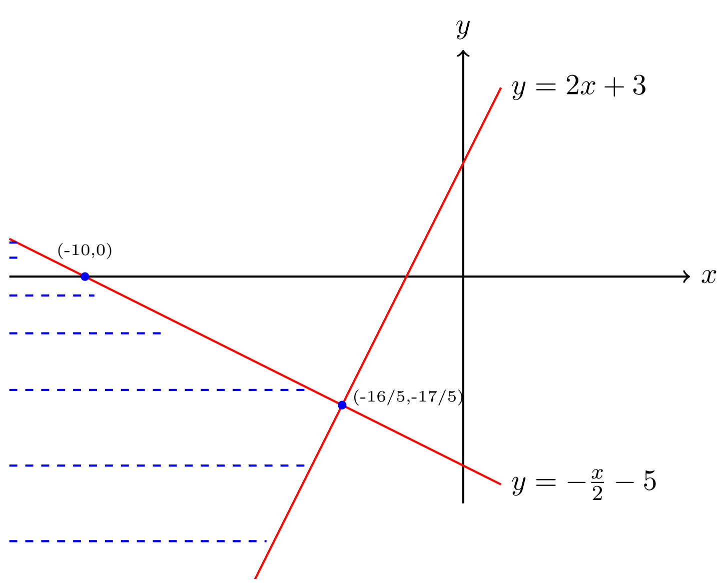
ANSWER: 1

Solution: The graph of $(x+5)^2+(y-12)^2 = 14^2$ is the circle of radius 14 with center $(-5, 12)$. Note that the origin $(0, 0)$ is inside the circle and for every point (x, y) on the circle, $x^2 + y^2$ is the square of the distance from (x, y) to $(0, 0)$. The shortest distance from (x, y) to $(0, 0)$ occurs when (x, y), $(0, 0)$ and $(-5, 12)$ are all on a radius of the circle. Since the distance from $(-5, 12)$ to $(0, 0)$ is 13 and the radius has length 14 then the distance from (x, y) to $(0, 0)$ is 1.

16. The xy-plane is divided into four quadrants: I, II, III and IV. If the point (x, y) satisfies $2x + 3 < y < -\frac{x}{2} - 5$, in what quadrant(s) could (x, y) be?

ANSWER: II and III

Solution: The point (x, y) lies above the line $y = 2x + 3$ and below the line $y = -\frac{1}{2}x - 5$. A rough sketch of the two lines is



17. What is the largest integer n such that $\frac{n^2-38}{n+1}$ is an integer?

ANSWER: 36

Solution:

$$\begin{aligned}\frac{n^2-38}{n+1} &= \frac{n^2-1-37}{n+1}\\ &= \frac{n^2-1}{n+1} - \frac{37}{n+1}\\ &= n-1-\frac{37}{n+1}.\end{aligned}$$

So $\frac{37}{n+1}$ must be an integer and the largest value for n is $n = 36$.

18. The six digit number $3730A5$, where A is the tens digit, is divisible by 21. Find all possible values of A.

ANSWER: 6

Solution: Since $3730A5$ is divisible by 21, it is divisible by both 3 and 7. Since it is divisible by 3, 3 must divide $3+7+3+0+A+5 = A+18$. Thus, 3 must divide A. Hence $A \in \{0, 3, 6, 9\}$. Dividing by 7,

$$\begin{array}{r@{\,}l} & 532 \\ 7 & \overline{)3730A5} \\ & \underline{35} \\ & 23 \\ & \underline{21} \\ & 20 \\ & \underline{14} \\ & 6A5 \end{array}$$

we see that 7 must divide $6A5$ with $A \in \{0, 3, 6, 9\}$.

$$\begin{aligned} 700 - 605 &= 95 \qquad A = 0\\ 700 - 635 &= 65 \qquad A = 3\\ 700 - 665 &= 35 \qquad A = 6\\ 700 - 695 &= 5 \qquad A = 0. \end{aligned}$$

So 7 divides 665 and does not divide the other three. Hence $A = 6$.

19. How many pairs of integers (x, y) satisfy the equation $y = \frac{x+12}{2x-1}$?

ANSWER: 6

Solution: We seek the number of integers x such that $\frac{x+12}{2x-1}$ is an integer.

$$\frac{x+12}{2x-1} = \frac{1}{2}\frac{2x+24}{2x-1} = \frac{1}{2}\frac{2x-1+25}{2x-1} = \frac{1}{2}(1 + \frac{25}{2x-1}).$$

In order for $\frac{x+12}{2x-1}$ to be an integer, $\frac{25}{2x-1}$ must be an odd integer. The possibilities are $x = 1, 3, 13$ and $x = 0, -2, -12$. Six in all.

20. Among all of the points (x, y) on the line $2x + 3y = 6$, find the value of x that gives the smallest value of $\sqrt{x^2 + y^2}$.

ANSWER: $\frac{12}{13}$

Solution: The problem essentially asks for the point (x, y) on the line so that the distance from (x, y) to the origin is a minimum. That same point has the property that $x^2 + y^2$ is a minimum. We have

$$\begin{aligned} x^2 + y^2 &= x^2 + (2 - \tfrac{2}{3}x)^2 \\ &= \tfrac{13}{9}x^2 - \tfrac{8}{3}x + 4. \end{aligned}$$

Completing the square gives

$$x^2 + y^2 = \tfrac{13}{9}(x - \tfrac{12}{13})^2 + 4 - \tfrac{12}{9}.$$

This is clearly a minimum when $x = \frac{12}{13}$.

21. All of the positive integers are written in a triangular pattern beginning as follows and continuing in the same way:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & 3 & 4 & & \\ & 5 & 6 & 7 & 8 & 9 & \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{array}$$

Which number appears directly below 2014?

ANSWER: 2104

Solution: Notice that row n ends in n^2 and begins with $(n-1)^2 + 1$. Since $44^2 < 2014 < 45^2$, 2014 is in row 45. The entry below an entry in row n is obtained by adding $2n$. In our case $n = 45$ and the entry below 2014 is $2014 + 90 = 2104$.

22. Daisy has twenty 3¢ stamps and twenty 5¢ stamps. Using one or more of these stamps, how many different amounts of postage can she make?

ANSWER: 152

Solution: Twenty 5¢ stamps has the same monetary value as thirty 3¢ stamps with two 5¢ stamps. It is easy to see that the postage amounts for twenty 3¢ and twenty 5¢ is the same as that for fifty 3¢ and two 5¢. The postage amounts for the latter are

50 using only 3¢ stamps,
51 using none or more 3¢ stamps with exactly one 5¢ stamp,
51 using none or more 3¢ stamp with exactly two 5¢ stamps.
The total is $50 + 51 + 51 = 152$.

23. How many triples (x, y, z) of rational numbers satisfy the following system of equations?

$$x + y + z = 0$$
$$xyz + z = 0$$
$$xy + yz + xz + y = 0.$$

ANSWER: 2

Solution: Assume $z = 0$. Then

$$x + y = 0$$
$$xy + y = 0.$$

Since $y = -x$ then $-x^2 - x = 0$ so $x = 0$ or $x = -1$. $x = 0$ gives the solution $(0, 0, 0)$. $x = -1$ gives the solution $(-1, 1, 0)$.

Assume $z \neq 0$, then

$$z = -(x + y)$$
$$xy = -1$$

and $xy + (x + y)z + y = 0$. The last equation becomes

$$-1 - (x + y)^2 + y = 0$$
$$y = x^2 + y^2 - 1.$$

Using $xy = -1$, i.e. $y = -\frac{1}{x}$ we have $-\frac{1}{x} = x^2 + \frac{1}{x^2} - 14$ or $x^4 - x^2 + x + 1 = 0$. This last equation has $x = -1$ as a solution, which does not lead to a new solution of the original problem. Dividing $x^4 - x^2 + x + 1$ by $x + 1$ we get $x^3 - x^2 + 1 = 0$. By the rational root theorem the last equation has no rational roots.