BC Exam Solutions
Texas A&M High School Math Contest
November 8, 2014

If units are involved, include them in your answer, which needs to be simplified.

1. If an arc of 60° on circle I has the same length as an arc of 45° on circle II, what is the ratio of the area of circle I to the area of circle II?

   Solution: Let r_1 and r_2 denote the radii of circles I and II respectively. Then we have

$$r_1 \frac{\pi}{3} = r_2 \frac{\pi}{4} \implies \frac{r_1}{r_2} = \frac{3}{4} \implies \frac{A_1}{A_2} = \frac{\pi r_1^2}{\pi r_2^2} = \left(\frac{r_1}{r_2}\right)^2 = \frac{9}{16}.$$

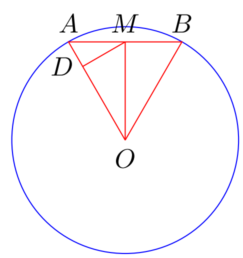
2. How many positive integers are there, which are less than 1000 and not divisible by 5 or 7?

   Solution: There are 199 integers less than 1000 divisible by 5 and 142 less than 1000, which are divisible by 7. There are 28 integers less than 1000, which are divisible by 35. Thus, there are

$$999 - 199 - 142 + 28 = 686,$$

   integers less than 1000 not divisible by 5 or 7.

3. In a circle of radius r centered at O a chord AB of length r is drawn. From O a perpendicular to AB meets AB at M. From M a perpendicular to OA meets OA at D. In terms of r, what is the area of triangle MDA?



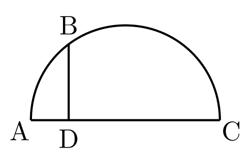
   Solution: Note that ΔOAB is an equilateral triangle. Thus, $\overline{AM} = r/2$, $\overline{OM} = \sqrt{3}r/2$, and the area of

$$\Delta AMO = \frac{\sqrt{3}r^2}{8}.$$

   ΔAMD is similar to ΔOAM, with a scaling coefficient of $1/2$. Thus, the area of ΔAMD is $1/4$ that of ΔOAM. Thus,

$$\text{Area of } \Delta MDA = \frac{\sqrt{3}r^2}{32}.$$

4. The figure below is a half circle with $AD = 3$ inches and $DC = 12$ inches. Line BD is perpendicular to the half circles diameter AC and intersects the circle at point B. How long is line BD?



   Solution: Let x denote the length of line BD. Then

$$\begin{aligned} 15^2 &= AB^2 + BC^2, \quad 3^2 + x^2 = AB^2, \quad 12^2 + x^2 = BC^2 \\ 15^2 &= (3^2 + x^2) + (12^2 + x^2) \\ 2x^2 &= 15^2 - 12^2 - 3^2 = 72 \\ x &= 6 \end{aligned}$$

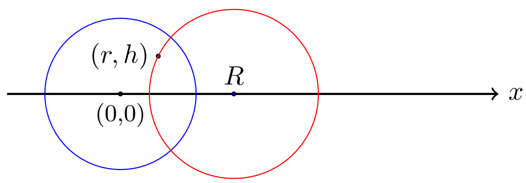
5. The first 50 terms of an arithmetic series sum to 200 and the next 50 terms sum to 2700. What is the value of the first term?

Solution: The k^{th} term of the series is $a_k = a + (k-1)d$, and

$$\begin{aligned} 200 &= \sum_{k=1}^{50} a_k = \sum_{k=1}^{50}(a+(k-1)d) = 50a + \frac{49*50}{2}d \\ 2700 &= \sum_{k=51}^{100} a_k = \sum_{k=1}^{100} a_k - \sum_{k=1}^{50} a_k = 50a + (99*50 - \frac{49*50}{2})d \\ 2500 &= 2500d \implies d = 1\,. \end{aligned}$$

Solving either of the first two equations for a we get $a = -20.5$.

6. The figure below shows two intersecting circles. The circle to the left is centered at the origin and has radius 1. The second circle is centered at $(R, 0)$, and it intersects the unit circle orthogonally, i.e., perpendicularly. The second circle also passes through the point (r, h), which is interior to the unit circle. Find R in terms of r and h.



Solution: Denote the coordinates of the point of intersection in the upper half plane by (x, y). We have the following relations:

$$\begin{aligned} x^2 + y^2 &= 1, \text{ the point } (x,y) \text{ is on unit circle,} \\ (x-R)^2 + y^2 &= (r-R)^2 + h^2, \text{ the points } (x,y) \text{ and } (r,h) \text{ are on the circle} \\ \frac{y}{x-R} &= -\frac{x}{y}, \text{ since} \end{aligned}$$

the line from the origin to (x, y) is perpendicular to the line from $(R, 0)$ to (x, y). The first and third equations imply $xR = 1$ Expanding the second equation and using the fact that $xR = 1$ solve for R and get

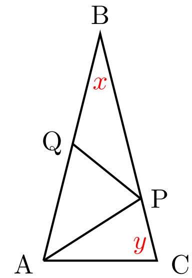
$$R = \frac{1 + r^2 + h^2}{2r}\,.$$

7. Suppose the polynomial $p(x) = x^{2014} - c_1x^{2013} + c_2x^{2012} + \cdots + c_{2014}$ has roots $\{\pm1, \pm2, \cdots, \pm1007\}$. What do c_1 and c_{2014} equal?

Solution: $c_1 = 0$ as c_1 is the sum of the roots, and $c_{2014} = -\left[1007!\right]^2$, as it is the product of the roots.

8. Triangle ABC is isosceles with base AC. Points P and Q are respectively in CB and AB such that $AC = AP = PQ = QB$. What are the number of degrees in angle B?

   Solution:



   Let x equal $\angle ABC$ and y equal $\angle ACB$. Then, since triangles ABC, PAC, BQP and APQ are all isosceles, we have

$$x + 2y = \pi, \quad 2x + \angle BQP = \pi, \quad 2y + \angle PAC = \pi, \quad 2x = \angle AQP = \angle PAQ \implies y - 2x = \angle PAC\,.$$

   These equations imply

$$x + 2y = \pi, \quad 3y - 2x = \pi\,.$$

   Solving for x we have $x = \frac{\pi}{7} = \frac{180}{7}$ degrees.

9. Thirty one books are arranged from left to right in increasing order of price, and the price of each book differs by \$2.00 from each adjacent book. Moreover, the price of the most expensive book equals the sum of the prices of the middle book and a book adjacent to the middle book. What is the price of the most expensive book?

   Solution: Let p_i, for $i = 1, \cdots, 31$, denote the price of each book. Then we have $p_i = 2(i-1) + p_1$. We know that one of the following two equations is true:

$$\begin{aligned} p_{31} &= p_{16} + p_{15}, &\quad \text{or} \quad & p_{31} = p_{16} + p_{17} & (1)\\ 2\cdot 30 + p_1 &= (2\cdot 15 + p_1) + (2\cdot 14 + p_1), &\quad \text{or} \quad & 2\cdot 30 + p_1 = (2\cdot 15 + p_1) + (2\cdot 16 + p_1) & (2)\\ 60 + p_1 &= 58 + 2p_1, &\quad \text{or} \quad & 60 + p_1 = 62 + 2p_1 & (3) \end{aligned}$$

   The first of equations (3) implies $p_1 = 2$, while the second of equations (3) implies $p_1 = -2$. Thus, $p_1 = 2$, and the price of the most expensive book is

$$p_{31} = 60 + 2 = 62$$

10. The function $p(x) = ax^2 + bx + c$ describes a parabola. Suppose the parabola's vertex is located at the point $(-1, 2)$. What does b/a equal?

    Solution: Completing the square we have

$$p(x) = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \left(c - \frac{b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

    The ordinate of the vertex occurs when the quadratic term is zero. Thus, $-1 + \frac{b}{2a} = 0$, or $\frac{b}{a} = 2$.

11. Jane can mow a field in 12 hours, while Jane and Bill working together can mow the field in 8 hours. How long will it take Bill to mow the field by himself?

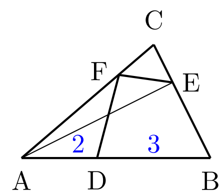
    Solution: In 8 hours Jane will have mowed $2/3^{\text{rd}}$ of the field. Thus, Bill will have mowed $1/3^{\text{rd}}$ of the field, which means that it would take Bill 24 hours to mow the field by himself.

12. The probability that event A occurs is $7/8$, and the probability that event B occurs is $5/6$. What is the smallest possible value of the probability of $A \cap B$?

Solution: We know that $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$. Since $\Pr(A \cup B) \leq 1$, we have

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) \geq \frac{7}{8} + \frac{5}{6} - 1 = \frac{17}{24}\,.$$

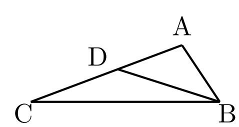
13. Triangle ABC in the figure below has area 10. Points D, E, and F all distinct from the vertices of the triangle lie on sides AB, BC, and CA respectively; AD=2 and DB =3. If triangle ABE and quadrilateral DBEF have equal areas, what is that area?



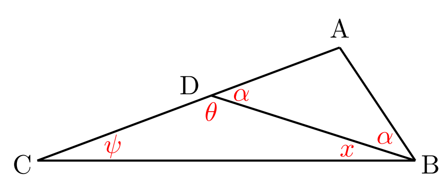
Solution: Notice that triangle DBE is part of both triangle ABE and quadrilateral DBEF. Thus, the area of ABE = area of AED + area of DBE, and area of DBEF = area of DFE + area of DBE. Hence triangles ADE and FDE have the same area. Moreover, since they have a common base they must have the same altitudes from that base, which means that points A and F are equidistant from the line through D and E. Thus, the line through A and F is parallel to the line through D and E, which implies that triangles ABC and DBE are similar. Thus, the ratio of their altitudes must equal $5/3$. The altitude of triangle ABC from the base AB equals 4. Thus, the altitude of triangle DBE $=4 \cdot 3/5 = 12/5$. Triangles ABE and DBE have equal altitudes, which means that the area of triangle ABE equals

$$\frac{1}{3} \cdot 5 \cdot \frac{12}{5} = 4\,.$$

14. In ΔABC, a point D in on AC so that $AB = AD$. If $\angle ABC - \angle ACB = \pi/6$, what does $\angle CBD$ equal?



Solution: Since $AD = AB$, angles ADB and ABD are equal. Denote the common value by α. Let ψ denote the value of angle DCB, $\theta = \pi - \alpha$ the value of angle CDB, and x the value of angle CBD.



We have the following equations:

$$\alpha + x - \psi = \frac{\pi}{6}, \quad \psi + (\pi - \alpha) + x = \pi\,.$$

Adding these two equations we have $2x + \pi = \pi + \pi/6$, which implies that

$$x = \frac{\pi}{12} \text{ or 15 degrees.}$$

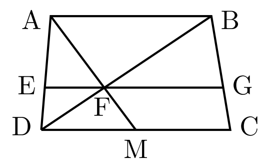
15. Daisy has twenty 3¢ stamps and twenty 5¢ stamps. Using one or more of these stamps, how many different amounts of postage can she make?

Solution: Twenty 5¢ stamps has the same monetary value as thirty 3¢ stamps with two 5¢ stamps. It is easy to see that the postage amounts for twenty 3¢ and twenty 5¢ is the same as that for fifty 3¢ and two 5¢. The postage amounts for the latter are

50 using only 3¢ stamps,
51 using none or more 3¢ stamps with exactly one 5¢ stamp,
51 using none or more 3¢ stamp with exactly two 5¢ stamps.

The total is $50 + 51 + 51 = 152$.

16. In the figure below



ABCD is a trapezoid with AB and DC parallel. AM is a median of ΔADC, DB is a diagonal of the trapezoid with the median AM meeting the diagonal DB at F. Line EG passes through F and is parallel to DC. If ΔAEF has area equal to 6 sq. cm., what is the area of ΔBFG?

Solution: Note that ΔAEF is similar to ΔADM and that ΔBFG is similar to ΔBDC. Moreover, since the three horizontal lines are parallel we have

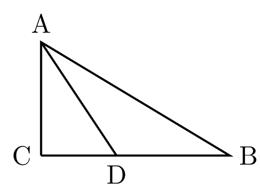
$$\frac{AE}{AD} = \frac{BG}{BC}\,.$$

Since we also have

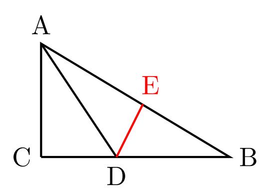
$$\frac{EF}{DM} = \frac{AE}{AD} = \frac{BG}{BC} = \frac{GF}{CD}, \text{ and } CD = 2DM\,.$$

we may conclude that $GF = 2EF$. Since the triangles AEF and BGF have the same altitudes we may conclude that the area of ΔBGF is twice that of ΔAEF, so the area of ΔBGF is 12 sq. cm.

17. Triangle ABC has a right angle at C, $AC = 2$, and
$BC = 3$. The bisector of $\angle BAC$ meets BC at D. Find CD.



Solution: Let DE be the altitude of $\triangle ADB$.

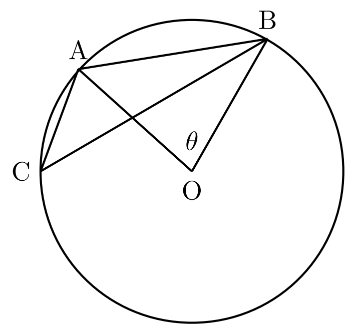


Then note that $\triangle ACD$ is congruent to $\triangle AED$, and so $AE = AC = 2$. Then $AB = \sqrt{13}$. Let $CD = x$. Then $DE = x, EB = \sqrt{13} - 2$, and $DB = 3 - x$. Applying the Pythagorean Theorem to $\triangle DEB$ yields $x^2 + (\sqrt{13} - 2)^2 = (3 - x)^2$, from which $x = \frac{4\sqrt{13}-8}{6} = \frac{2}{3}(\sqrt{13} - 2)$.

18. A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. Find the ratio of the number of cubes with exactly two green faces to the number of cubes with exactly three green faces.

    Solution: The cubes with two green faces are the cubes along the edges, not counting the corner cubes. In each dimension, we lost two cubes to the corners so we then have four edges with 4 cubes, four with 3 cubes and four with 2 cubes. The total number of cubes with paint on two faces is then $4 \times 4 + 3 \times 4 + 2 \times 4 = 36$. The number of cubes that have paint on three sides are the corner cubes of which there are eight. The required ratio is then $36 : 8$ or $9 : 2$.

19. The circle shown below is centered at O, has radius equal to 1, and $\theta = 24°$. What is the sum of angles ACB and OAB?



    Solution: Angle $\angle ACB = \theta/2$, and $\angle OAB = (\pi-\theta)/2$. Thus the sum of these two angles is

$$\angle ACB + \angle OAB = \frac{\theta}{2} + \frac{(\pi - \theta)}{2} = \frac{\pi}{2}.$$