Best Student Exam - Open Section
Texas A&M High School Math Contest
8 November 2014

1. The numbers 1, 2, 3, etc. are written down one after the other up to 1000, without any commas, in one long string that begins

$$123456789101112131415161718192021222\cdots,$$

and ends with 9989991000. How many 0s appear in this string?

2. How many real numbers $x$ are there with $0 \le x \le 2\pi$ so that $\sin(10\cos x)$ belongs to the set $\{-1, -1/2, 0, 1/2, 1\}$?

3. What is the area of the largest circle that can be inscribed in a triangle with edges of length 5, 7, and 8?

4. How many ways are there to scramble the letters of the word "contest"? That is, how long is the list that starts with "cenostt" and ends with 'ttsonec" and includes "contest" and "settcon" along the way, as well as every other way to order the letters?

5. Find, and simplify fully, $\frac{d}{dx}\log(\sec(x)+\tan(x))$. Here, log means the natural logarithm, base $e$, not base 10.

6. An equilateral triangle is inscribed in the ellipse that is the graph of $x^2/4+y^2=1$, with one vertex at $(0,1)$. Find the height of the triangle.

7. Find $(\sqrt{3}+i)^{12}$ (in fully simplified form).

8. How many real numbers $y$ are there so that

$$\sin(y) = \frac{1}{3}y^{2/3}?$$

9. Take as given for this problem and the next two that the graph of the natural logarithm function, $x \to \log(x)$, is concave down, and that, in particular, it lies below any of its tangent lines. Find the unique number $c$ so that for all $x > -1$, $\log(1+x) \le cx$.

10. True or false: For all $x > -1$, $\log(1+x+x^2/4) \le cx$ (where $c$ is the correct value for the problem above.)

11. Find the largest number $d$ with the property that (for reasons like the one for the previous statement) $\log(1+x+dx^2) \le x$ for all $x \ge 0$.

12. How many ways are there to place four identical rooks on a chessboard (that is, on a square grid, 8 by 8) so that none of the rooks attacks another? (Equivalently, how many ways are there to put four pennies on the board in such a way that no two pennies are in the same row or the same column?)

13. Two random numbers $x$ and $y$ are drawn independently from the closed interval $[0, 2]$. What is the probability that $x + y > 1$?

14. Find the remainders $r_5$, $r_{25}$, and $r_{125}$ when $2^{32}$ is divided by 5, 25, and 125 respectively.

15. Find the smallest positive integer $n$ so that the remainder when $2^n$ is divided by 125 is 1.

16. Two silvered mirrors each pass $1/4$ of the light striking them head on, and reflect $3/4$ of it.

    They're set parallel to each other, and a beam of light is directed head-on at the pair. Some of the light that arrives at the first mirror passes through it, bounces around between the two for zero or one or any number of reflections, and eventually passes through the second mirror. What fraction of the original light eventually gets through to the other side of both mirrors?

17. Now there are three such mirrors. What fraction of the original light eventually gets through all three?

18. A version of rock-paper-scissors involves players Amy and Jan, who must simultaneously and secretly declare either 'heads' or 'tails'. Jan scores one point if their choices match, either HH or TT. Amy scores 1 point when Jan picks H and Amy picks T.

    If Jan picks T and Amy picks H, though, Amy scores *2* points.

    As partial compensation for this advantage, Jan gets to know Amy's strategy in advance. Amy may pick any number $a$ with $0 \leq a \leq 1$ and have a random number generator choose to declare 'H' for her if a random number generator that picks numbers at random in the interval $[0, 1]$ picks a number less than $a$, and declares 'T' for her otherwise. After that, Amy can only watch. She cannot adjust for how Jan plays.

    With this in mind, and Amy knows that Jan is clever and will figure out what $a$ is and play accordingly, Amy must pick $a$. What is her best choice of $a$, so as to be likely to score the most net points in the long run even though she knows Jan will be playing as well as possible and will know $a$?