Best Student Exam (Open and Closed) Solutions
Texas A&M High School Math Contest
8 November 2014

1. The numbers 1, 2, 3, etc. are written down one after the other up to 1000, without any commas, in one long string that begins

$$123456789101112131415161718192021 22\cdots,$$

and ends with 9989991000. How many 0s appear in this string?

The answer is 192. Two ways to organize the count suggest themselves. In both cases we begin by attaching 0 to the list and removing 1000. That can be set right at the end.

In the first way, imagine that the numbers have been written in a 3 by 1000 array instead of as one long string. The first column would be 000, the second column would be 001, the 477th column would be 476, and so on out to the last column, 999. In this setup, every digit occurs equally often and there would be 300 zeros. But some of those zeros don't occur in a listing that suppresses leading zeros. All the zeros in the top row go away. That leaves us with 200 zeros. The first ten zeros in the middle row go away, the ones corresponding to the first zero in 00 through 09. That leaves us with 190 zeros. The first zero in the bottom row goes away because we didn't start our list with 0, but with 1. That leaves us 189 zeros. Now we throw in three more because 1000 *is* in our list, and the answer is 192.

In the second approach, we first consider the numbers 1 through 9. No zeros there. We then consider the numbers 10 through 99. Nine zeros there. We finally consider the numbers 100 through 999. One in every ten numbers is a multiple of 10, which gives us 90 zeros, all in the final place. In the middle place, we have zeros at X0Y, where X can take 9 values and Y can take 10 values. That gives us another 90 zeros, bringing the count to 189. And finally, again there are the zeros in 1000, bringing the overall count to 192.

2. How many real numbers x are there with $0 \le x \le 2\pi$ so that $\sin(10\cos x)$ belongs to the set $\{-1, -1/2, 0, 1/2, 1\}$? There are 54, 27 on either side of π. The graph of $\cos x$ is symmetric about π—that is, $\cos(\pi + x) = \cos(\pi - x)$. Thus $\sin(10\cos x)$ is also symmetric about π.

Now $\sin(10\cos \pi) = \sin(-10) = -\sin(10) = \sin(10 - 3\pi)$ which is approximately $\sin(0.58)$ (no better approximation than 3.14 for π is needed here). Now $\sin(\pi/6) = 1/2$ and the sine function is increasing on $(0, \pi/2)$ so when it works out that $.57 > \pi/6$, it follows that $1/2 < \sin(-10) < 1$.
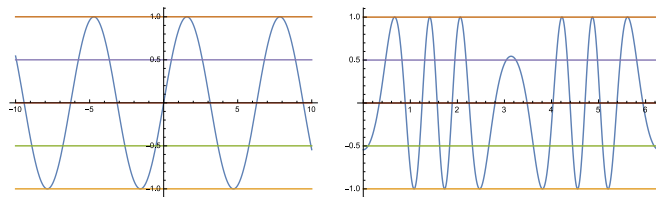
As x ranges up from π to 2π, $\cos x$ increases from -1 to 1, and $10\cos x$ increases from -10 to 10. For the same reason that $\sin(-10)$ lies between $1/2$ and 1, $\sin(10)$ lies between -1 and $-1/2$. The graph of $\sin y$, plotted against benchmarks at $-1, -1/2, 0, 1/2, 1$, includes 3 complete periods plus excursions on both ends to beyond $\pm 1/2$. The three full periods yield seven zeros, six each of $\pm 1/2$, and three each of ± 1 for a total of 25 further points to count. That comes to 27 total for half the graph, and thus 54 for the whole graph.

3. What is the area of the largest circle that can be inscribed in a triangle with edges of length 5, 7, and 8? The answer is 3π.

   The required circle is called the *incircle*, and it fits inside the triangle so that the line segment from its center to each edge of the triangle makes a right angle with that edge. Call its center the incenter, and its radius, r.

   Cut the big triangle into three smaller triangles using the incenter as a vertex of each of the new triangles. The areas of the three pieces are $5r/2$, $7r/2$, and $8r/2$, so the area A of the original triangle is given by $A = 10r$. On the other hand, the area of the original triangle is also given by Heron's formula as $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ and the sides have lengths a, b, and c. Here, $s = 10$ so $A = \sqrt{300} = 10\sqrt{3}$. Thus $10\sqrt{3} = 10r$ and $r = \sqrt{3}$ so the area of the incircle is 3π.

4. How many ways are there to scramble the letters of the word "contest"? That is, how long is the list that starts with "cenostt" and ends with 'ttsonec" and includes "contest" and "settcon" along the way, as well as every other way to order the letters?

   If the two t's came in different fonts, the answer would be simpler. There are 7 ways to say where the first T goes. There are then six ways to say where the second t goes. Then 5 ways to say where the s goes, and so on. That would lead to an answer of $7! = 5040$. But each pair of scrambles that are alike except for which font goes with which t condenses into one scramble when we remove the font distinction from the t's. Thus the answer is 2520.

5. Find, and simplify fully, $\frac{d}{dx}\log(\sec(x) + \tan(x))$. Here, log means the natural logarithm, base e, not base 10.

   The answer is $\sec x$. Start by recalling, or working out starting with the quotient rule and knowledge of the calculus of sin and cos, that $\sec x' = \tan x \sec x$ and $\tan x' = \sec^2 x$. Then factor out $\sec x + \tan x$ from the fraction that results from the rule $\log(u(x))' = u'(x)/u(x)$, itself a consequence of the chain rule and the fact that $\log x' = 1/x$.

6. An equilateral triangle is inscribed in the ellipse that is the graph of $x^2/4 + y^2 = 1$, with one vertex at $(0, 1)$. Find the height of the triangle.

The other two vertices have to be on the bottom part of the ellipse, and symmetrical placed across the x axis, say at $(-a, -b)$ and $(a, -b)$, where a and b are positive and $a^2/4 + b^2 = 1$.

For the triangle to be equilateral, we need $2a = \sqrt{a^2 + (b+1)^2}$. That leads to $3a^2 = (b+1)^2$. On the other hand, $3a^2 + 12b^2 = 12$, so $12 - 12b^2 = (b+1)^2$. The values for b that work are given by the quadratic formula. One of them is -1, which we don't want, and the other is $b = 11/13$. The height of the triangle is thus $H = 24/13$.

7. Find $(\sqrt{3} + i)^{12}$ (in fully simplified form). One way to work this is to recognize that $u\sqrt{3} + i = 2(cos\pi/6) + i\sin\pi/6) = 2v$, say. Now by De Moivre's formula $(\cos x + i \sin x)^n = \cos nx + i \sin nx$, we have that $u^{12} = 2^{12}v^{12}$ and $v^{12} = 1$ because 12 times 30 degrees is 360. Now $2^{12} = 4096$ because $2^6 = 64$ and $64^2 = 4096$. Another way would be to multiply and get values for $u^2 = 2 + 2\sqrt{3}i$, then $u^3 = 8i$, then $u^6 = -64$, and from there also, $u^{12} = 4096$.

8. How many real numbers x are there so that

$$\sin(x) = \frac{1}{3}x^{2/3}?$$

There are five. The graphs of the two expressions $\sin(x)$ and $(1/3)x^{2/3}$ look something like this:
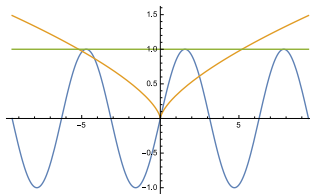
The only question is how soon does it happen that the 'gull wings' of the non-trigonometric expression flap above 1 and out of reach of the sine function.

Here, we just need two arithmetic facts: $(1/3)(3\pi/2)^{2/3} < 1$, and $(1/3)(2\pi)^{2/3} > 1$. Now it can be difficult without a calculator to evaluate these numbers to sufficient accuracy, but the inequality can be tested by taking both sides to the third power. So now it comes down to $\pi^2 < 12$ for the one, and $(2\pi)^2 > 27$ for the other. Both are easy given that $3 < \pi < 3.2$, which is well known.

9. Take as given for this problem and the next two that the graph of the natural logarithm function, $x \to \log(x)$, is concave down, and that, in particular, it lies below any of its tangent lines.

Find the unique number c so that for all $x > -1$, $\log(1 + x) \le cx$. The derivative of the log function at $x = 1$ is 1, so the slope of the line tangent

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at $(0,0)$ to the graph of $\log(1+x)$ is 1. Any line through $(0,0)$, other than the tangent line, will nick the graph on either its right or its left. So $c = 1$.

10. True or false: For all $x > -1$, $\log(1+x+x^2/4) \leq cx$ (where c is the correct value for the problem above.) This is true because $1+x+x^2/4 = (1+x/2)^2$ and $\log(1+x/2)^2 = 2\log(1+x/2) \leq 2x/2 = x$ (from the above).

11. Find the largest number d with the property that (for reasons like the one for the previous statement) $\log(1 + x + dx^2) \leq x$ for all $x \geq 0$. Think about $\log((1 + x/n)^n)$? The same logic applies and all these are $\leq x$ for the same reason. Now $(1 + x/n)^n$ expands with the binomial theorem as $1 + nx/n + (n(n-1)/2)x^2+$ more stuff, all of which will be positive if x is positive. It follows that for $x > 0$,

$$\log\left(1 + x + \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right)x^2\right) \leq x.$$

The largest value of d that works in this argument is $d = 1/2$.

12. How many ways are there to place four identical rooks on a chessboard (that is, on a square grid, 8 by 8) so that none of the rooks attacks another? (Equivalently, how many ways are there to put four pennies on the board in such a way that no two pennies are in the same row or the same column?)

There are $\binom{8}{4} = 70$ ways to choose which set of four rows contains a rook. There are then as many ways to choose which set of four columns contains a rook. Now we're up to 4900 ways and we still haven't decided where, on the resulting four by four sub-board, the rooks go. There are 4! ways to do that, so the final answer is $4900 \times 24 = 117600$.

13. Two random numbers x and y are drawn independently from the closed interval $[0, 2]$. What is the probability that $x + y > 1$?

Equivalently, a single random point with coordinates (x, y) is drawn from the square $0 \leq x, y \leq 2$. This square has area 4. The region where $x+y < 1$ is the triangle bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$. That triangle has vertices $(0,0)$. $(1,0)$, and $(0,1)$ and it has area $1/2$. So the probability of the event's complement is $1/8$ and the required probability is $7/8$.

14. Find the remainders r_5, r_{25}, and r_{125} when 2^{32} is divided by 5, 25, and 125 respectively. If we had r_{125} we could get the others. Now $2^2 = 4$, $2^4 = 16$, and $2^8 = 256 \equiv 6 \pmod{125}$. So then $2^{16} \equiv 36 \pmod{125}$, and finally, $2^{32} \equiv 36^2 = 1296 \equiv 46 \pmod{125}$. The answers are thus 1, 21, and 46.

15. Find the smallest positive integer n so that the remainder when 2^n is divided by 125 is 1. From the previous problem, $2^4 \equiv 16 \pmod{125}$. It

is easily checked that for $1 \leq n \leq 3$, 2^n is not 1 mod 5, and the sequence goes into a loop after hitting 1, so we need only look for the answer among numbers n of the form $n = 4k$. Mod 25, these go 16, 6, 21, 11, and 1, (and then cycle) so our answer will actually lie among numbers of the form $n = 20k$. Now mod 125, $2^{20} \equiv 76$ since $2^{16} \equiv 36$ and $2^4 \equiv 16$ and $16 \times 36 = 576$. So, we have to find the first j so that $2^{20j} \equiv 1 \pmod{125}$, or equivalently, we want $(1+75)^j \equiv 1 \pmod{125}$. But $(1+75)^j \equiv 1+75j$ mod 125, so we just need $j = 5$. The least value of n is 100.

16. Two silvered mirrors each pass 1/4 of the light striking them head on, and reflect 3/4 of it.

    They're set parallel to each other, and a beam of light is directed head-on at the pair. Some of the light that arrives at the first mirror passes through it, bounces around between the two for zero or one or any number of reflections, and eventually passes through the second mirror. What fraction of the original light eventually gets through to the other side of both mirrors?

    Well, 1/4 of it gets in between the mirrors, and that 1/4 is split by the next mirror so that 1/16 goes through immediately, while 3/16 heads back toward the first mirror.

    That 3/16 is split into 3/64 that returns toward the light source, and 9/64 that heads back in the 'right' direction.

    We are now in the exact same position that we were when our first 1/4 of the light got through the first mirror. In one back-and-forth, our fraction 1/4 has turned into 9/64, a ratio of 9/16. In the next pass, we'll have $(1/4)(9/16)^2$ worth of light heading in the right direction and between the mirrors. The total of all that light is found by summing the geometric series
    $$(1/4)(1 + (9/16) + (9/16)^2 + \cdots) = \frac{1/4}{1 - 9/16} = \frac{4}{7}.$$

    Each time, 1/4 of what arrives at the last mirror, going the right way, goes through, so in the end, 1/7 of the light passes the array and 6/7 is ultimately reflected.

17. Now there are three such mirrors. What fraction of the original light eventually gets through all three?

    Think of the three mirrors as amounting to one thick mirror that passes 1/7 of the light and reflects 6/7, with the other passing 1/4 and reflecting 3/4. The amount of light that gets through the first, thick, mirror is 1/7. Of that, 1/28 gets through the second mirror right away, while 3/28 is reflected back toward the thick mirror, and so this time the light coming at the final mirror resolves into an initial portion of 1/7 plus follow on portions each times 9/14 compared to the previous portion. The total

arriving at the final mirror is thus

$$\frac{1}{7} \cdot \frac{1}{1 - 9/14} = \frac{2}{5}.$$

Since only 1/4 of what hits the final mirror gets through, the grand total that gets through the array of three mirrors is 1/10 of the original beam. (The other 9/10 is reflected.)

18. A version of rock-paper-scissors involves players Amy and Jan, who must simultaneously and secretly declare either 'heads' or 'tails'. Jan scores one point if their choices match, either HH or TT. Amy scores 1 point when Jan picks H and Amy picks T.

If Jan picks T and Amy picks H, though, Amy scores *2* points.

As partial compensation for this advantage, Jan gets to know Amy's strategy in advance. Her strategy must be to pick a number a with $0 \le a \le 1$ and have a random number generator declare 'H' for her if it outputs a number less than a, and 'T' otherwise. After that, Amy can only watch. She cannot adjust for how Jan plays.

With this in mind, and Amy knows that Jan is clever and will figure out what a is and play accordingly, Amy must pick a. What is her best choice of a, so as to be likely to score the most net points in the long run even though she knows Jan will be playing as well as possible and will know a?

Amy's best choice is to compute, wearing Jan's hat so to speak, how she would play if she were Jan, and then pick the best result from those possibilities.

The expected net point value to Amy of Amy's strategy, (to pick heads randomly with probability a, else tails) if Jan picks heads with probability b, is $-ab - (1-a)(1-b) + 2a(1-b) + (1-a)b = 3a + 2b - 5ab - 1$. If Amy chooses $a = 1/2$, for instance, then Amy's payoff is $2b - (5/2)b + 1/2 = (1-b)/2$. Since Jan wants the points to go to her and not Amy, she will want this expression to be as negative as possible. Jan will choose $b = 1$ if Amy chooses $a = 1/2$, and Amy's expected payoff will be 0.

The coefficient in front of b is key here. If it is negative, Jan will choose $b = 1$, while if it is positive, Jan will choose $b = 0$. The coefficient is $2 - 5a$. So if $a > 2/5$, the coefficient is negative and $b = 1$ and the expected point payoff to Amy works out to $1 - 2a$. If $a < 2/5$ then the coefficient is positive, $b = 0$, and the expected point payoff to Amy works out to $3a - 1$.

Amy's situation is that her expected payoff is the function of a given for $a \le 2/5$ by $3a - 1$ and for $a > 2/5$ by $1 - 2a$. This function is piecewise linear, increasing on $[0, 2/5]$ and decreasing on the rest of the interval. Amy's best choice is to take $a = 2/5$. As it happens, with $a = 2/5$, Jan's expected payoff is 1/5 no matter how she chooses b, and it simply won't matter what Jan's strategy is; Amy will net 1 point on average for each five passes of the game.

There is a whole field built around games of chance and strategy in the vein of this one: game theory. 6