CD EXAM SOLUTIONS
Texas A&M High School Math Contest
Oct 24, 2015

1. Consider an equilateral triangle ABC with sides of length 1. Extend AB from A to A' so that A lies between A' and B, and the distance from A' to A is x. Similarly extend AC to C' and CB to B'. Triangle $A'B'C'$ will also be equilateral. Find the smallest positive integer value for x so that the sides of $A'B'C'$ have integer lengths.

   If one sets up a coordinate system with $(0,0)$ and $(1,0)$ as two vertices of the original equilateral triangle, and $(1/2, \sqrt{3}/2)$ as the third, say, with one choice of labels it works out that A' has coordinates $(-x,0)$ and B' has coordinates $((x+1)/2,(x+1)\sqrt{3}/2)$. The square distance between these points is then $3x^2+3x+1$ and this needs to be an integer. Case by case examination shows that taking $x=7$ gives a square value, 169, for $3x^2+3x+1$.

   If we wanted more choices for x, there's a better way to find them than just trying possibilities one by one: rewrite the equation $3x^2+3x+1=m^2$ as $(2m)^2-3(2n+1)^2=1$. This is a special case of $u^2-3v^2=1$ and these arise out of expanding powers of $2-\sqrt{3}$ into the form $a-b\sqrt{3}$. The same power of $2+\sqrt{3}$ will then be $a+b\sqrt{3}$ (give or take a $\pm$ that disappears when we fix the parity of the power) and so $a^2-3b^2=1$.

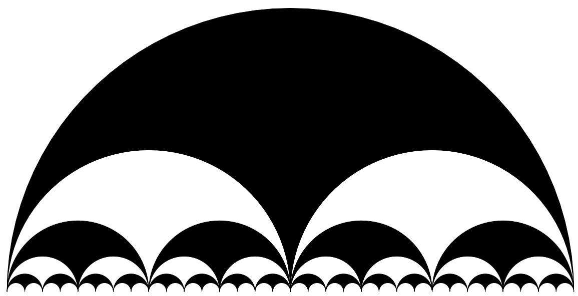
2. Find
$$\sqrt{2^1+\sqrt{2^2+\sqrt{2^4+\sqrt{2^8+\cdots}}}}$$
   and give the answer in the form $(a+\sqrt{b})/\sqrt{c}$ where a, b, and c are positive integers.

   Let x be the answer. Then
$$x/\sqrt{2}=\sqrt{1+\sqrt{+\sqrt{1+\cdots}}}=\sqrt{1+x/\sqrt{2}}.$$
   Solve the quadratic equation $x^2/2-x/\sqrt{2}-1=0$ to get $x/\sqrt{2}=(1+\sqrt{5})/2$. Thus $x=(1+\sqrt{5})/\sqrt{2}$.

3. Find positive integers a and b such that for all positive integers k,
$$10^{2(ak+b)}+10^{ak+b}+9$$
   is divisible by 7.

   We calculate mod 7. We have $100\equiv 2 \pmod 7$ and $10\equiv 3 \pmod 7$. So $10^{2(ak+b)}\equiv 2^{ak+b} \pmod 7$, $10^{ak+b}\equiv 3^{ak+b} \pmod 7$, and $9\equiv 2 \pmod 7$. Note also that $2^3\equiv 1 \pmod 7$ while $3^6\equiv 1 \pmod 7$. So if we take $a\equiv 6 \pmod 7$, whatever happens at one value of k happens at any other, just what we need. So we tentatively take $a = 6$ and look for a suitable value of b. We may as well take $k = 0$ (the expression takes the same value mod 7 for any k, after all) and this leads to the requirement that $2^b + 3^b + 2 \equiv 0 \pmod 7$. Taking $b = 1$ works. We could also take $b = 7, 13, 19, \ldots$, and we could take $a = 12, 18, 24, \ldots$. But out of these, $a = 6$, $b = 1$ is minimal. The answer is $a = 6$, $b = 1$.

4. A half circle of radius 1 has two half circles inscribed along its base, each of radius $1/2$. These have further half-circles inscribed, the pattern continuing to all depths.



What is the sum of the areas of the black-colored portions? The two half-disks that are removed each have one quarter the area of the original, so the first black region has area half the original. The next four each have 1/32nd the area of the original, and the next 16 each have area 1/512th the original area. The original area of the half-disk was $\pi/2$ so the geometric series has first term $\pi/4$ and common ratio $1/4$. Its sum is thus $\pi/3$.

5. A dartboard of radius r has zones bounded by circles of radius $r/4$, $r/2$, and $3r/4$, by the x and y axes, and by the lines $y = \pm x$. What is the farthest distance between two points in a single zone? There are two contenders for this distance. The point furthest to the right is $(r, 0)$. The point on the line $y = x$ at the edge of the board is $r(1/\sqrt{2}, 1/\sqrt{2})$, and these two are further apart than the second of them and $(3r/4, 0)$. The maximal distance is $r\sqrt{2 - \sqrt{2}}$.

6. A polynomial $p(z)$ is called *suitable* if it has the form

$$\begin{aligned} p(z) &= (z - w_1)(z - w_2)(z - w_3)(z - w_4) \\ &= (z - u_1 - iv_1)(z - u_2 - iv_2)(z - u_3 - iv_3)(z - u_4 - iv_4) \end{aligned}$$

where each of the u_j and v_j's is an integer. Find a suitable polynomial $p(z)$ such that $p(0) = 4$, $p(1) = 5$, and for all z, $(z^2 + 2z + 2)p(z - 2) = (z^2 - 6z + 10)p(z)$.

Note that

$$\begin{aligned} &(z + 1 + i)(z + 1 - i)(z - w_1 - 2)(z - w_2 - 2)(z - w_3 - 2)(z - w_4 - 2) = \\ &= (z - 3 + i)(z - 3 - i)(z - w_1)(z - w_2)(z - w_3)(z - w_4). \end{aligned}$$

This suggests that perhaps $w_1 = -1 - i$, $w_2 = -1 + i$, and if that is so, then after cancelling a factor of $z^2 + 2z + 2$ from both sides we have

$$(z-3-i)(z-3+i)(z-w_3-2)(z-w_4-2) = (z-3-i)(z-3-i)(z-w_3)(z-w_4).$$

This now suggests that perhaps $w_3 = 1+i$, $w_4 = 1-i$, and plugging those in works. Multiplying out $(z+1-i)(z+1-i)(z-1-i)(z-1+i)$ gives $p(z) = z^4 + 4$.

7. For which values of c does the system of equations

$$xy = 2/c, \qquad x^2 + y^2 = c$$

have four distinct real solutions? The points on $xy = K$ nearest the origin are $\pm(\sqrt{K}(1,1)$, at a distance of $\sqrt{2K}$ from the origin. For the hyperbola to meet the circle at all, we must have $\sqrt{2 \cdot 2/c} \leq \sqrt{c}$. That happens when $c \geq 2$. To have four intersections, we must have $c > 2$.

8. A man has \$ 1.70 in nickles, dimes, and quarters. He lists how many of each he has, producing a list of the form (n, d, q). (It might read $(4, 0, 6)$, say, or $(2, 1, 6)$, or $(0, 17, 0)$.) How many possibilities are there for this list? If there are 6 quarters, the other 20 cents can include 0, 1, or 2 dimes so there's 3 lists. If there are 5 quarters, the number of dimes can be anywhere from 0 to 4, so there are 5 lists. If 4 quarters, there are 8 choices for the number of dimes. If 3, there are 10 choices. If 2, there are 13 choices, if 1, there are 15 choices, and if he has no quarters, there are 18 choices for the number of dimes. All told, there are 72 possible lists.

9. A three dimensional chess board has 512 cubical 'squares'. A queen, on this board, can move in any direction a regular queen could move, in any of the three 8 by 8 planes that include here square. She can also move along any of the long diagonals through her position. What is the maximum number of squares such a queen can get to in one move from any particular position on the board? Put the queen at the 'square' (4,4,5) for best results. In any of the plane three 'boards' through her position, she has 13 'short diagonal' moves. Moving like a rook, she can go up-down, east-west, or north-south. Each of these three choices yields 7 more moves. We're up to $39 + 21 = 60$ possible moves. Then there are the long diagonals. There are four pairs of corners. Along the four long diagonals she can reach 6, 7, 6, and 6 spots for a total of 25 more moves. All told, she can reach 85 of the 511 squares of this board not counting the one she starts on.

10. Find $(\log_2 3)(\log_9 16)$. The rule is that $\log_a b = \ln b / \ln a$. Now $(\ln 3/\ln 2) \cdot (\ln 16/\ln 9)$ simplifies because $\ln 9 = 2\ln 3$ and $\ln 16 = 4\ln 2$. So the answer is 2.

11. Find the sum of the coordinates of all integer points strictly inside the triangle with vertices $(0, 0)$, $(20, 8)$, and $(21, 9)$. The answer is 68. There

are three points: $(12,5)$, $(17,7)$, and $(19,8)$. If (x,y) is a point of the sort required, then we must have $2/5 < y/x < 3/7$ and $x < 20$. The fractions between two rationals a/b and c/d such that $bc-ad=1$ are generated (with increasing denominators) by inserting between two adjacent fractions p/q and r/s the fraction $(p+r)/(q+s)$. Thus we get

$$\frac{2}{5} < \frac{5}{12} < \frac{3}{7}$$
$$\frac{2}{5} < \frac{7}{17} < \frac{5}{12} < \frac{8}{19} < \frac{3}{7}$$

and no more may be inserted because the denominator would exceed 20.

12. An equilateral triangle with vertices $(-1,0,0)$, $(0,1,0)$, and $(0,0,\sqrt{3})$is rotated around the z axis, forming a cone. A sphere is inscribed in the cone, tangent at a point on the x-y plane and tangent to the curved surface of the cone along a circle. A plane parallel to the x-y plane is also tangent to the sphere at its top. This cuts off a little cone at the top of the big cone. Find the ratio of the volume of the little cone to the volume of the big cone.

    The sphere cuts the x-z plane in a circle that is inscribed in the equilateral triangle with base $(\pm 1, 0)$ and height $\sqrt{3}$. The center of that triangle is $(0, 1/\sqrt{3})$ and the radius is $1/\sqrt{3}$. Thus the height of the small cone is $\sqrt{3} - 2/\sqrt{3} = 1/\sqrt{3}$. The small cone thus has $1/3$ the height of the large one, but the same shape. That means its volume is $1/27$th the volume of the big cone. The ratio is $1/27$.

13. Given that $x^2 - y^2 = 2$ and $x^3 - y^3 = 3$, find $x + y - (xy/(x+y))$. The answer is $3/2$. We have

$$x + y - \frac{xy}{x+y} = \frac{x^2+xy+y^2}{x+y}$$
$$= \frac{(x-y)(x^2+xy+y^2)}{(x-y)(x+y)} = \frac{x^3-y^3}{x^2-y^2} = \frac{3}{2}.$$

14. Let

$$S = \sum_{k=1}^{2015} \frac{1}{k(k+1)(k+2)} = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{2015\cdot 2016\cdot 2017}.$$

    The decimal expansion of S has the form $a.bcdefghijkl\cdots$. Find a, b, c, and d. ANSWER: $a=0$, $b=2$, $c=4$, and $d=9$. We begin with the observation that $1/(a(a+b)) = 1/b(1/a - 1/(a+b))$. Using this twice on each term in the original sum yields

$$S = \frac{1}{2}\sum_{k=1}^{2015} \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}.$$

This can be rewritten as

$$S = \frac{1}{2}\left(\sum_{k=1}^{2015}\left[\frac{1}{k} - \frac{1}{k+1}\right] - \sum_{k=1}^{2015}\left[\frac{1}{k+1} - \frac{1}{k+2}\right]\right).$$

Each bracketed sum telescopes. The first telescopes to a bit less than 1 while the second telescopes to a (smaller) bit less than 1/2. Thus S is a bit less than 1/4, and the decimal expansion of S starts off $S = 0.249...$ and so $a = 0$, $b = 2$, $c = 4$, and $d = 9$.

15. How many copies of 2 appear in the prime factorization of $100!/(50!)^2$? Of the numbers 1 to 100, 50 are even. Of these, 25 are even again. Of these, 12 are even again, and so on. In all, we get $50 + 25 + 12 + 6 + 3 + 1 = 97$ powers of 2 in 100!. Similarly, there are $25 + 12 + 6 + 3 + 1 = 47$ powers of 2 in 50!, so subtracting 94 powers of 2 that occur in the denominator from the 97 in the numerator, we arrive at the answer: 3.

16. Find all ordered triples of integers (a, b, c) so that $|a + b| + |b + c| = 1$ and $|a + b| + |a + c| = 3$.

    If $a = 0$, then $|b| + |b + c| = 1$ and $|b| + |c| = 3$. Thus $|c| - |b + c| = 2$. So $|c| \geq 2$ and b is even. But if $b = 0$ we get a contradiction $1 = 3$, while if $|b| \geq 2$ we get the contradiction that $1 = |b| + |b + c| \geq 2$. So $a \neq 0$.

    If $|a| = 1$ then $|a + c| - |b + c| = 2$ so b is odd. If $|b| \geq 3$ then $|a + b| \geq 2$ and we have another contradiction, so $|b| = 1$. Together with $|a + b| \leq 1$ we conclude $b = -a$. We now need $|c - a| = 1$ and $|c + a| = 3$. There are two ways this can happen: $\pm(1, -1, 2)$.

    If $a = 2$, there are two sub-cases that stand a chance of giving something: $b = -1$ and $b = -2$. If $b = -1$ we need $|-1 + c| = 0$ so that $c = 1$ and then $|a + b| + |a + c| = 4$. We cannot take $b = -1$ after all. If $a = 2$ and $b = -2$, we must take $c = 1$. By similar logic, $(-2, 2, -1)$ provides the fourth and final solution. The whole list is $\pm(1, -1, 2)$ and $\pm(2, -2, 1)$.

17. Find the least prime p that divides $10^{10^{10}} + 10^{10} + 10 - 1$. Clearly $p = 2$ does not work. If $p = 3$, then the two smaller terms sum to 9 but the last two are powers of ten and hence each are 1 mod 3, so the whole expression is 2 mod 3. Clearly 5 does not work.

    If $p = 7$ then any power of 10 is congruent to 4 mod 6, so as in a previous problem, for any positive integer a, $10^(10^a) \equiv 10^4 \pmod 7 \equiv 3^4 \equiv 4$. But $4 + 4 + 9$ is not a multiple of 7, so 7 doesn't work either. What about $p = 11$? Any odd power of 10 is congruent to -1 mod 11, and any even power of 10 is congruent to $+1$. So, and with a nod of thanks to the William Lowell Putnam competition in which a related problem was posed, the number we say is divisible by 11 is congruent, mod 11, to $1 + 1 - 1 - 1$. That is, it's divisible by 11 as claimed.