DE Exam, Solutions
Texas A&M High School Math Contest
October 24, 2015

Answers should include units when appropriate.

1. Let t_1 and t_2 be the times from start till the boats meet for the first and the second time, respectively. If w is the width of the river, and v is the sum of speeds of the boats, then $vt_1 = w$ and $vt_2 = 3w$, hence $t_2 = 3t_1$. One ferry traveled distance 700 feet at time t_1, and at time t_2 it traveled $w + 400 = 3 \cdot 700$ feet. Therefore, $w = 3 \cdot 700 - 400 = 1700$ feet.

2. Let a, b, and c be productivities of the workers (job per hour). Then the statements of the problem are:

$$\frac{1}{a+b+c} = \frac{1}{a} - 6 = \frac{1}{b} - 1 = \frac{1}{2c},$$

and we are asked to find $\frac{1}{a+b}$.

We have from $\frac{1}{a+b+c} = \frac{1}{2c}$ that $a + b = c$. We get then

$$\frac{a+b}{a} - 6(a+b) = \frac{1}{2}; \qquad \frac{a+b}{b} - (a+b) = \frac{1}{2},$$

or

$$1 + \frac{b}{a} - 6c = \frac{1}{2}; \qquad 1 + \frac{a}{b} - c = \frac{1}{2}.$$

Writing $\frac{a}{b}$ and $\frac{b}{a}$ in terms of c, and using $\frac{a}{b} \cdot \frac{b}{a} = 1$, we get

$$\left(6c - \frac{1}{2}\right)\left(c - \frac{1}{2}\right) = 1,$$

which leads to the quadratic equation

$$6c^2 - \frac{7}{2}c - \frac{3}{4} = 0.$$

Its roots are $\frac{\frac{7}{2} \pm \sqrt{49/4+18}}{12} = \frac{\frac{7}{2} \pm \sqrt{121/4}}{12} = \frac{7 \pm 11}{24}$. Discarding the negative root, we get $c = 3/4$, hence $\frac{1}{a+b} = \frac{1}{c} = 4/3$ hours $= 80$ minutes.

3. We can rewrite $\sqrt{n} - \sqrt{n-1} = \frac{n-(n-1)}{\sqrt{n}+\sqrt{n-1}} = \frac{1}{\sqrt{n}+\sqrt{n-1}}$, hence we are asked to find the smallest integer n such that $\sqrt{n} + \sqrt{n-1} > 100$. If $\sqrt{n} + \sqrt{n-1} > 100$, then $2\sqrt{n} > 100$, hence $n > 2500$. If we take $n = 2501$, then $\sqrt{n} + \sqrt{n-1} > 2\sqrt{n-1} = 100$. Therefore, the answer is $n = 2501$.

4. We have $|x + y| \le |x| + |y|$, hence the sum is less than 3. On the other hand, two of the numbers, say x and y, are of the same sign. Then $|x + y| = |x| + |y|$, and one of the summands is equal to 1. Hence, the sum S satisfies $1 \le S \le 3$. If we set $x = 1, y = 1, z = -a$, where $0 < a \le 1$, then the expression is equal to $1 + 2\frac{1-a}{1+a}$. The function $\frac{1-a}{1+a}$ takes all values in the interval $[0, 1)$ when $0 < a \le 1$. Therefore, our original expression takes all values in the interval $[1, 3)$ for $x = y = 1, z = -a$. The value 3 is attained for $x = y = z = 1$. It follows that the set of possible values of the expression is the closed interval $[1, 3]$.

5. From the second equation $y - 2xy = 0$ we get $y(1 - 2x) = 0$, hence $y = 0$ or $x = 1/2$. If $y = 0$, then the first equation becomes $x = x^2$, which has solutions $x = 0$ or $x = 1$. If $x = 1/2$, then the first equation is $1/2 = 1/4 + y^2$, hence $y^2 = 1/4$, which has solutions $y = 1/2$ or $y = -1/2$. Therefore the answer is $(x, y) = (0, 0), (1, 0), (1/2, 1/2)$, or $(1/2, -1/2)$.

6. Since $0 \le x < \pi$, $\sin x$ is positive, and we can write the first equation as

$$\sqrt{1 - \cos^2 x} + \cos x = \frac{1}{5},$$

or

$$\frac{1}{5} - \cos x = \sqrt{1 - \cos^2 x}.$$

Taking square of both sides, we get

$$\frac{1}{25} - \frac{2}{5}\cos x + \cos^2 x = 1 - \cos^2 x,$$

or

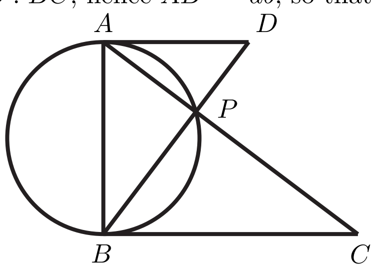
$$2\cos^2 x - \frac{2}{5}\cos x - \frac{24}{25} = 0.$$

Solving it as a quadratic equation for $\cos x$, we get

$$\cos x = \frac{\frac{2}{5} \pm \sqrt{\frac{4}{25} + \frac{8\cdot 24}{25}}}{4} = \frac{\frac{2}{5} \pm \frac{2}{5}\sqrt{1+48}}{4} = \frac{1 \pm 7}{10},$$

hence $\cos x = 4/5$ or $\cos x = -3/5$. Then $\sin x = \sqrt{1-\cos^2 x}$ is either $\sqrt{1-16/25} = 3/5$ or $\sqrt{1-9/25} = 4/5$, respectively. It follows that $\tan x = 3/4$ or $\tan x = -4/3$.

7. Let P be the intersection point of AC and BD. Since P is on the circle and AB is a diameter, $\angle APB = 90°$. It follows that $\angle DBA = 90° - \angle CAB = \angle ACB$, hence $\triangle DAB$ is similar to $\triangle ABC$. Then $DA : AB = AB : BC$, hence $AB^2 = ab$, so that the diameter of the circle is $\sqrt{ab}$.



8. Let us assume that $a \le b$. The case $b \le a$ will be similar. If $-a \le x \le a$, then $-b \le x \le b$, and $|x-a| + |x+a| = 2a$, $|x-b| + |x+b| = 2b$, hence for $m = 2$ solutions are all numbers $-a \le x \le a$.

If $a \le x \le b$, then $|x-b| + |x+b| = 2b$, and $|x-a| + |x+a| = x + a + x - a = 2x$, so that the equation becomes $2x + 2b = m(a+b)$, which has solution $x = \frac{m}{2}a + \frac{m-2}{2}b$. It must satisfy $a \le x \le b$, which is equivalent to $2a + 2b \le 2x + 2b \le 4b$, which is equivalent to $2(a+b) \le m(a+b) \le 4b$, i.e., $2 \le m \le \frac{4b}{a+b}$.

If $x \ge b$, then $|x-a| + |x+a| = 2x$ and $|x-b| + |x+b| = 2x$, so that we get $x = \frac{m(a+b)}{4}$, which must satisfy $\frac{m(a+b)}{4} \ge b$, which is equivalent to $m \ge \frac{4b}{a+b}$.

The cases $-b \le x \le -a$ and $x \le -b$ are reduced to the previous cases by replacing x by $-x$ in the equation.

It follows that the set of values for which the equation has a solution is $[2, +\infty)$.

9. We have $\cos 2\alpha = 2\cos^2\alpha - 1$, hence $\cos^2\alpha = \frac{1+m}{2}$, and $\sin^2\alpha = 1 - \frac{1+m}{2} = \frac{1-m}{2}$.

Using the identity $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$, we get $\sin^6\alpha + \cos^6\alpha = (\cos^2\alpha + \sin^2\alpha)(\cos^4\alpha - \cos^2\alpha\sin^2\alpha + \sin^4\alpha) = \cos^4\alpha - \cos^2\alpha\sin^2\alpha + \sin^4\alpha = \frac{(1+m)^2}{4} - \frac{1-m^2}{4} + \frac{(1-m)^2}{4} = \frac{1+2m+m^2-1+m^2+1-2m+m^2}{4} = \frac{3m^2+1}{4}$.

10. The remainder is a polynomial of the form $ax + b$ satisfying

$$x^{2015} = P(x)(x^2 - 3x + 2) + (ax + b)$$

for some polynomial $P(x)$. The polynomial $x^2 - 3x + 2$ has roots 1 and 2, therefore

$$1 = a + b, \qquad 2^{2015} = 2a + b$$

Subtracting the first equation from the second, we get $a = 2^{2015} - 1$. Then, from the first equation we get $b = 2 - 2^{2015}$. Therefore, the answer is $(2^{2015} - 1)x + (2 - 2^{2015})$.

11. We have $\log_y x = \frac{1}{\log_x y}$. Denoting $\log_x y = t$, we get from the first equation of the system

$$\frac{1}{t} + t = 5/2,$$

hence

$$t^2 - \frac{5}{2}t + 1 = 0,$$

which has solutions

$$t = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{\frac{5}{2} \pm \frac{3}{2}}{2},$$

hence $t = 2$ or $t = 1/2$. If $t = 2$, then $\log xy = 2$, i.e., $y = x^2$. If $t = 1/2$, then $x = y^2$. In the first case the second equation gives $x^3 = 27$, in the second case we get $y^3 = 27$. Hence, the answer is $(x, y) = (3, 9)$ or $(9, 3)$.

12. We have $\sqrt[3]{0.5} + \sqrt[3]{4} = \frac{1}{\sqrt[3]{2}} + \sqrt[3]{4} = \frac{1+2}{\sqrt[3]{2}} = \frac{3}{\sqrt[3]{2}}$. The equation becomes

$$\frac{3^x}{2^{x/3}} = \frac{27}{2}.$$

It is easy to see that $x = 3$ is a solution. It is unique, since the function $\left(3/\sqrt[3]{4}\right)^x$ is increasing.

13. Let us replace $\sin(xy)$ by another variable a. The quadratic equation $x^2 + 2ax + 1 = 0$ in x has discriminant $D = 4a^2 - 4$. Since x has to be real, $D \geq 0$. But $-1 \leq a \leq 1$, so $a = \pm 1$. Hence, we have two cases:

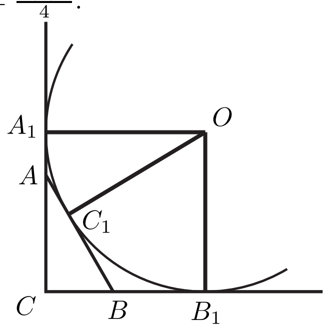
$$\begin{cases} \sin(xy) &= 1 \\ x^2 + 2x + 1 &= 0 \end{cases}$$

and

$$\begin{cases} \sin(xy) &= -1 \\ x^2 - 2x + 1 &= 0 \end{cases}$$

In the first case we get $x = -1$ and $y = -\pi/2 + 2k\pi$, $k \in \mathbb{Z}$. In the second case we get $x = 1$ and $y = -\pi/2 + 2k\pi$, $k \in \mathbb{Z}$.

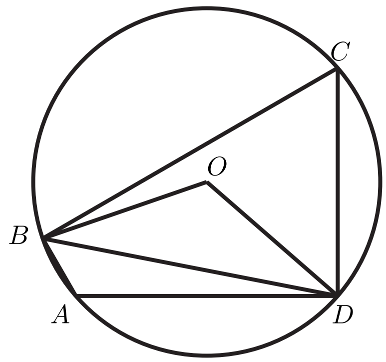
14. Let A_1, B_1, and C_1 be the feet of the perpendiculars from the center O of the circle to the lines AC, CB, and AB, respectively. Then OA_1CB_1 is a square. Denote the radius by r. Then $BB_1 = r - 1/2$, $AA_1 = r - \sqrt{3}/2$. We have $BB_1 = BC_1$ and $AA_1 = AC_1$. Hence, $(r - 1/2) + (r - \sqrt{3}/2) = 1$. It follows that $r = \frac{3+\sqrt{3}}{4}$.



15. Since B and D are right angles, we can circumscribe a circle around $ABCD$, and AC will be its diameter. Let O be the center of the circle, and r its radius. Then $\angle DOB = 120°$, since $\angle DCB = 60°$. Using the law of cosines, we get

$$DB^2 = 46^2 + 13^2 + 2 \cdot 46 \cdot 13 \cdot \frac{1}{2} = 2116 + 169 + 598 = 2883 = 3 \cdot 961 = 3 \cdot 31^2.$$

In the triangle DOB we have $OD = OB = r$, $DB = 31\sqrt{3}$, $\angle DOB = 120°$. It follows that $DB = 2 \cdot r \cdot \frac{\sqrt{3}}{2} = 31\sqrt{3}$, hence $r = 31$, and $AC = 62$.



16. In other words, we are asked to solve the system

$$\begin{cases} cx^3 - x^2 - x - (c+1) & = & 0 \\ cx^2 - x - (c+1) & = & 0 \end{cases}$$

We have $cx^3 - x^2 = x + c + 1 = cx^2$, hence $x^2(cx - 1 - c) = 0$. It follows that either $x = 0$, or $cx - 1 - c = 0$. In the first case we get $c + 1 = 0$, hence $c = -1$. In the second case we have $c \neq 0$, and $x = \frac{1+c}{c}$. Substituting $x = \frac{1+c}{c}$ into the equations, we see that both of them are satisfied. It follows that the polynomials have a common root for all $c \neq 0$. Note that for $c = 0$ the polynomials are $-x^2 - x - 1$ and $-x - 1$, so they have no common roots. Answer: all $c \neq 0$.

17. We have $\sin 3x = \sin x \cos 2x + \sin x \cos 2x = \sin x \cos 2x + 2 \sin x \cos^2 2x$. Therefore, the equation is equivalent to

$$\sin x(\cos 2x + 2\cos^2 x - 2) = 0.$$

If $\sin x = 0$ and $0 \leq x < 2\pi$, then $x = 0$ or π. If $\sin x \neq 0$, then

$$\cos 2x + 2\cos^2 x - 2 = 0$$

which can be written

$$2\cos^2 x - 1 + 2\cos^2 x - 2 = 0,$$

or $4\cos^2 x = 3$, hence $\cos^2 x = 3/4$, or $\cos x = \pm\sqrt{3}/2$, which gives solutions $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

18. The segment AD is the bisector of the angle BAE in triangle BAE. It is known (a corollary of the law of sines) that $AB : AE = BD : DE$. Denote $AB = x$, $AD = y$. Then $AE = 3x/2$. Similarly, using the fact that AE is the bisector of the angle DAC, we get $AD : AC = DE : EC$, hence $AC = 2y$. Let us denote $\cos\angle BAD = c$. Then, by the law of cosines, we get

$$\begin{cases} x^2 + y^2 - 2xya & = & 4 \\ \frac{9}{4}x^2 + y^2 - 3xya & = & 9 \\ \frac{9}{4}x^2 + 4y^2 - 6xya & = & 36 \end{cases}$$

Subtract the first equation from the second:

$$\frac{5}{4}x^2 - xya = 5, \qquad xya = \frac{5}{4}x^2 - 5.$$

Subtract the second equation from the third:

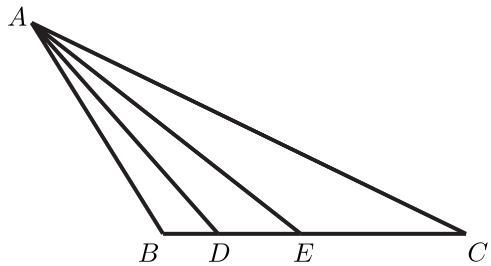
$$3y^2 - 3xya = 27, \qquad xya = y^2 - 9.$$

We get $y^2 - 9 = \frac{5}{4}x^2 - 5$, hence $y^2 = \frac{5}{4}x^2 + 4$. Substituting this and $xya = \frac{5}{4}x^2 - 5$ into the first equation, we get

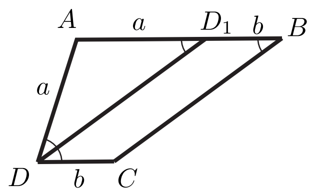
$$x^2 + \frac{5}{4}x^2 + 4 - \frac{5}{2}x^2 + 10 = 4,$$

$$\frac{1}{4}x^2 = 10$$

$$x = \sqrt{40}.$$



19. Let DD_1 be the bisector of the angle D, where D_1 is a point on the segment AB. Then $\angle AD_1D = \angle D_1DC = \angle ADD_1$, hence $\triangle DAD_1$ is isosceles, so that $AD_1 = a$. The line DD_1 is parallel to BC, hence DD_1BC is a parallelogram, so that $D_1B = DC = b$. We conclude that $AB = a + b$.



20. We have $f(n) + f(1) = f(n+1) - n - 1$, hence $f(n+1) = f(n) + n + 2$. It follows $f(2) = 1 + 3$, $f(3) = 1+3+4$, $f(4) = 1+3+4+5$, e.t.c., $f(n) = 1+3+4+5+\cdots+n+1 = \frac{(n+1)(n+2)}{2} - 2 = \frac{n^2+3n-2}{2}$.

Note that (replacing n by $n-1$) we get $f(n-1) = f(n) - n - 1$, which gives a proof by induction for the formula $f(n) = \frac{n^2+3n-2}{2}$ also for negative n.

We have to solve

$$\frac{n^2+3n-2}{2} = n,$$

$$n^2 + 3n - 2 = 2n$$

$$n^2 + n - 2 = 0.$$

Roots are $n = 1, -2$.

21. The condition implies that the polynomial $(x+1)P(x) - x$ has roots $0, 1, 2, \ldots, n$. Since $(x+1)P(x) - x$ has degree $n+1$, it follows that $(x+1)P(x) - x = cx(x-1)(x-2)\cdots(x-n)$ for some non-zero number c. Consequently, $P(x) = \frac{cx(x-1)(x-2)\cdots(x-n)+x}{x+1}$. Since $P(x)$ is a polynomial, -1 must be a root of the numerator $cx(x-1)(x-2)\cdots(x-n)+x$. It follows that $c(-1)(-2)(-3)\cdots(-n-1) - 1 = 0$, hence $c = (-1)^{n+1}/(n+1)!$, and $P(x) = \frac{\frac{(-1)^{n+1}}{(n+1)!}x(x-1)(x-2)\cdots(x-n)+x}{x+1}$. We get $P(n+1) = \frac{\frac{(-1)^{n+1}}{(n+1)!}(n+1)n(n-1)\cdots1+(n+1)}{n+2} = \frac{(-1)^{n+1}+n+1}{n+2}$. In other words, $P(n+1) = \frac{n}{n+2}$ if n is even, and $P(n+1) = 1$ if n is odd.