Solutions to Best Student Exam

Texas A&M High School Math Contest

22 October, 2016

1. The prime factors of 20! are 2, 3, 5, 7, 11, 13, 17, and 19, so $A = 77$. Likewise, the prime factors of 16! are 2, 3, 5, 7, 11, and 13, so $B = 41$. Therefore, $A + B = \mathbf{118}$.

2. At 5pm, the angle between the minute and hour hand is $\frac{5}{12}(360^\circ) = 150^\circ$. Each minute, the minute hand moves 6° clockwise, but the hour hand moves $\left(\frac{1}{60}\right)\left(\frac{1}{12}\right)(360^\circ) = \dfrac{1}{2}^\circ$ clockwise, so the hands move $5\frac{1}{2}^\circ$ closer together. After m minutes, the hands are at an angle of $150 - \dfrac{11}{2}m$. We want $150 - \dfrac{11}{2}m = 90$, or $m = \dfrac{120}{11} \approx 11$ minutes, so the time is about $\mathbf{5:11pm}$.

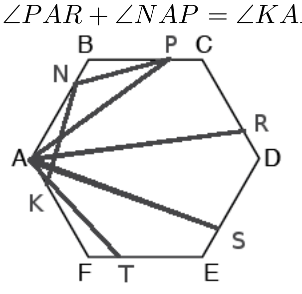
3. Remove one inch from the length, then divide the remaining block into one $15 \times 5 \times 7\frac{1}{2}$ inch block, which can be cut into 15 $3 \times 5 \times 2\frac{1}{2}$ inch pieces, and one $15 \times 3 \times 7\frac{1}{2}$ inch block, which can be cut into 9 $5 \times 3 \times 2\frac{1}{2}$ inch pieces. This gives us a total of **24 pieces.**

4. Let N be the number of coins each pirate receives at the end, meaning there are $3N$ coins in the chest. Working backwards, the third pirate discovers $\dfrac{3}{2}N + \dfrac{3}{2}N + \left(\dfrac{3}{2}N + 1\right) = \dfrac{9}{2}N + 1$ coins in the chest. The second pirate discovers $\left(\dfrac{9}{4}N + \dfrac{1}{2}\right) + \left(\dfrac{9}{4}N + \dfrac{1}{2}\right) + \left(\dfrac{9}{4}N + \dfrac{1}{2} + 1\right) = \dfrac{27}{4}N + \dfrac{5}{2}$ coins, and the first pirate discovers $\left(\dfrac{27}{8}N + \dfrac{5}{4}\right) + \left(\dfrac{27}{8}N + \dfrac{5}{4}\right) + \left(\dfrac{27}{8}N + \dfrac{5}{4} + 1\right) = \dfrac{81}{8}N + \dfrac{19}{4}$ coins. Since the number of coins must be a whole number, N must be even, so the smallest possible value is $N = 2$, leaving **25 coins** originally in the chest.

5. Note that the middle barrel is not used in any of the sides, so we place 9 there. The remaining barrels add up to 36, but note that the barrels on the vertices are added along two sides. We place 0 (the unmarked barrel), 1, and 2 along these sides, giving us a total of 39, divided equally among three sides with a sum of **13**.

6. It can be shown that the nth line segment adds $n + 1$ triangles to the existing ones (if D_n is the point of intersection with the segment and $\overline{AC}$, then triangles BAD_n, BCD_n, and BD_iD_n, for all $i < n$, are formed). Therefore, there are a total of $1 + 2 + 3 + \cdots + 2016 + 2017$ triangles from 2016 line segments. The sum is equal to $\dfrac{(2017)(2018)}{2} = (2017)(1009) = \mathbf{2,035,153}$ **triangles.**

7. Note that $FK = AN$. Select points P, R, S, and T on segments $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EF}$ such that $FK = AN = BP = CR = DS = ET$. (see figure below) Then $\angle KBN = \angle TAK$, $\angle KCN = \angle SAT$, $\angle KDN = \angle RAS$, $\angle KEN = \angle PAR$, and $\angle KFN = \angle NAP$. Therefore, $\angle KAN + \angle KBN + \angle KCN + \angle KDN + \angle KEN + \angle KFN = \angle KAN + \angle TAK + \angle SAT + \angle RAS + \angle PAR + \angle NAP = \angle KAN + \angle KAN = 120^\circ + 120^\circ = \mathbf{240}^\circ$.



8. Note that square of an integer is either 0 or 1 modulo 4. It follows that the only way the sum of three squares integers is divisible by 4 is that all three of them are divisible by 2. Consequently, if $(2n)^2 = a^2 + b^2 + c^2$ for integers n, a, b, c, then $n^2 = (a/2)^2 + (b/2)^2 + (c/2)^2$ and $a/2, b/2, c/2$ are integers. We have $2016 = 32 \cdot 63$, so if $2016^2 = a^2 + b^2 + c^2$ for positive integers a, b, c, then $63^2 = (a/32)^2 + (b/32)^2 + (c/32)^2$ and $a/32$, $b/32$, $c/32$ are integers. Denote $a_1 = a/32$, $b_1 = b/32$, $c_1 = c/32$. We want to find all triples of positive integers a_1, b_1, c_1 such that $a_1^2 + b_1^2 + c_1^2 = 63^2$. If we know that $a_1 \geq b_1 \geq c_1$, then a_1 is more than $\sqrt{63^2/3}$ and not more than 63, so it is between 37 and 63. We consider all these cases. In each of the cases there is an interval of possible values of b_1, and computing for each of them c_1, we get all possible triples:

38	37	34
43	38	26
46	37	22
48	33	24
48	39	12
49	28	28
50	37	10
50	38	5
53	26	22
53	34	2
54	27	18
56	28	7
57	24	12
58	22	11
59	22	2
60	15	12
62	11	2

To get the values of a, b, c, one has to multiply these numbers by 32. In particular, the smallest possible value of c is 64, but many other values are possible: 1088, 896, 832, 768, 704, 576, 384, 352, 320, 224, 160, 64.

9. The problem is equivalent to finding five numbers which, when one or more are added and/or subtracted (weights placed with the salt), yield any number up to 121. Proceed inductively as follows: start with a 1-ounce weight. Double and add one for a 3-ounce weight, allowing us to combine for up to 4 ounces $(1,\ 3-1,\ 3,\ 3+1)$. Double this and add one again for a 9-ounce weight, allowing us to combine for up to 13 ounces (append $9-4,\ 9-3,\ \cdots,\ 9,\ 9+1, \cdots 9+4$ to the previous list). Double this total and add one again for a 27-ounce weight, allowing us to combine for up to 40 ounces using the previous strategy. Double this total and add one again for an 81-ounce weight, allowing us to combine for up to 121 ounces. Mathematically, we can prove by induction that $2\left(\sum_{i=0}^{n} 3^i\right) + 1 = 3^{n+1}$. So our heaviest weight is **81 ounces.**

10. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{20}{1 - \tan\alpha\tan\beta}$. From the second given equation, $\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta} = \dfrac{\tan\beta + \tan\alpha}{\tan\alpha\tan\beta} = 16$, so $\tan\alpha\tan\beta = \dfrac{20}{16} = \dfrac{5}{4}$. Therefore, $\tan(\alpha+\beta) = \dfrac{20}{1 - \frac{5}{4}} = \mathbf{-80}$.

11. Let x be the number of cows, y be the number of sheep, and z be the number of rabbits. Then

$$\begin{gathered} x + y + z = 100 \\ 500x + 100y + 5z = 10000 \end{gathered}$$

Eliminate z to obtain $495x + 95y = 9500$. Since $95y$ and 9500 are both multiples of 19, $495x$ must also be a nonzero multiple of 19, meaning x is a nonzero multiple of 19. However, $x < 20$ because of the second equation above, so $x =$ **19 cows**.

12. Factor and apply properties of logarithms:

$$2\log_{5x+9}(x+3) + \log_{x+3}(5x+9) + \log_{x+3}(x+3) = 4$$

Let $t = \log_{5x+9}(x+3)$. From the change-of-base formula, $\log_{x+3}(5x+9) = \frac{1}{t}$, so our equation becomes $2t + \frac{1}{t} + 1 = 4$, or $2t^2 - 3t + 1 = 0$. Then $(2t-1)(t-1) = 0$, so $t = \frac{1}{2}$ or $t = 1$. Substitute back to obtain

$$\begin{array}{cc} \log_{5x+9}(x+3) = \frac{1}{2} & \log_{5x+9}(x+3) = 1 \\ x+3 = \sqrt{5x+9} & x+3 = 5x+9 \\ x^2+6x+9 = 5x+9 & -4x = 6 \\ x^2 + x = 0 & x = -\frac{3}{2} \\ x = -1 \text{ or } x = 0 & \end{array}$$

All three test as valid solutions; therefore, the smallest solution is $x = \mathbf{-\frac{3}{2}}$.

13. The water is in the shape of a cylinder with diameter 20cm and height 16cm containing a cylindrical "hole" with diameter 10cm and height 8cm. Therefore, the volume of water is $\pi(10^2)(16) - \pi(5^2)(8) = 1600\pi - 200\pi = 1400\pi$ cm^3. Let h be the original height of the water. Then $\pi(10^2)h = 1400\pi$, or $h =$ **14 cm**.

14. There are 24 different four-digit numbers possible, and each digit appears in each place value six times. If the original digits are a, b, c, and d, then $S = 6000(a+b+c+d) + 600(a+b+c+d) + 60(a+b+c+d) + 6(a+b+c+d) = (a+b+c+d)(6666) = (a+b+c+d)(2)(3)(11)(101)$. Therefore, the largest prime factor of S (regardless of the original digits) is **101**.

15. The function can be written as $f(x) = 2x + (x-1)^{-1} + (x+1)^{-1}$, so $f^{(n)}(x) = (-1)^n(n!)\left((x-1)^{-(n+1)} + (x+1)^{-(n+1)}\right)$. Therefore, $f^{(2016)}(2) = \mathbf{2016!\left(1+3^{-2017}\right)} = \mathbf{2016!\left(1+\frac{1}{3^{2017}}\right)}$.

16. Using properties of logarithms, $a_n = 2\ln(n) - \ln(n-1) - \ln(n+1)$. Then the sum of the numbers can be written as:

$$\begin{gathered} 2\ln(2) - \ln(1) - \ln(3) \\ +2\ln(3) - \ln(2) - \ln(4) \\ +2\ln(4) - \ln(3) - \ln(5) \\ +2\ln(5) - \ln(4) - \ln(6) \\ \cdots \end{gathered}$$

With cancellations, the sum after writing k lines is just $\ln(2) + \ln(k+1) - \ln(k+2) = \ln(2) + \ln\left(\frac{k+1}{k+2}\right)$. As k gets larger, this last term becomes very close to $\ln(1) = 0$, so the sum of an infinite number of these is simply $\ln(\mathbf{2})$.

17. With some rearranging, we find that

$$\begin{gathered}(x+y)(x^2-xy+y^2)+3xy-1=0\\(x+y)(x^2-xy+y^2)-(x^2-xy+y^2)+(x^2+2xy+y^2-1)=0\\(x+y-1)(x^2-xy+y^2)+(x+y-1)(x+y+1)=0\\(x+y-1)(x^2-xy+y^2+x+y+1)=0\end{gathered}$$

So our equation reduces down to $x+y-1=0$ or $x^2-xy+y^2+x+y+1=0$. But the second equation can be further rearranged to

$$\begin{gathered}\frac{1}{2}x^2-xy+\frac{1}{2}y^2+\frac{1}{2}x^2+x+\frac{1}{2}+\frac{1}{2}y^2+y+\frac{1}{2}=0\\\frac{1}{2}\left((x-y)^2+(x+1)^2+(y+1)^2\right)=0\end{gathered}$$

Which is only true at $(-1,-1)$. Therefore, our "curve" is simply this point along with the line $x+y=1$. To form a triangle, we must include the point $(-1,-1)$. The height h of our triangle is the distance from $(-1,-1)$ to $\left(\frac{1}{2},\frac{1}{2}\right)$, which is $\frac{3\sqrt{2}}{2}$. Then the length s of a side of our triangle is $\frac{2}{\sqrt{3}}h=\sqrt{6}$. Therefore, our area is $\frac{\sqrt{6}^2\sqrt{3}}{4}=\mathbf{\frac{3\sqrt{3}}{2}}$.