

CD EXAM
Texas A&M High School Math Contest
October 22, 2016

Directions: If units are involved, include them in your answer.

1. Let r be a real number such that

$$\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3.$$

Determine the value of

$$r^3 + \frac{1}{r^3}.$$

2. The front tires of a car will wear out after 15,000 miles, while the rear tires will wear out after 25,000 miles. When should we switch the front and rear tires to maximize the distance we can drive with the car?
3. We have made a regular triangular pyramid with 120 congruent spheres. How many spheres fit in the basis of the pyramid?
4. A non-constant real function f satisfies

$$f(x + y) = f(x) + f(y) - 2f(xy),$$

for all x, y . Find $f(2016)$.

5. Find $f\left(\frac{1}{14}\right) + f\left(\frac{2}{14}\right) + f\left(\frac{3}{14}\right) + \cdots + f\left(\frac{13}{14}\right)$, where $f(x) = \frac{4^x}{4^x + 2}$.
6. How many natural numbers n exist such that

$$1! + 2! + 3! + \cdots + n!$$

is a complete square?

7. An accurate analog clock has been wrongly designed in that its minute and hour hands are indistinguishable from each other. In the 12-hour period from noon till (but before) midnight, how many moments are there when it is not possible to immediately tell the time on this clock?
8. Find all real values of x such that

$$\left(\sqrt{4 - \sqrt{15}}\right)^x + \left(\sqrt{4 + \sqrt{15}}\right)^x = 8.$$

9. Find the sum of all positive numbers x such that $\sqrt[3]{2+x} + \sqrt[3]{2-x}$ is an integer.
10. What is the 2016th digit after the decimal point of the number $(\sqrt{31} + \sqrt{32})^{2016}$?

11. Two points A, B are 4 miles apart, and the midpoint between them is the center of a circular lake of diameter 2 miles. What is the shortest distance for an ant that wants to travel from A to B ?

12. Find the smallest number k such that the inequality

$$\left| \frac{a+b}{2} - \sqrt{ab} \right| \leq k|a-b|$$

holds for all positive numbers a, b .

13. Solve the equation $x^3 + x^2 + x = -\frac{1}{3}$.

14. A trapezoid $ABCD$ is isosceles, with $\overline{AD} = \overline{BC} = 2$, $\overline{AB} = 4$, and $\angle ABC = \angle BAD = 120^\circ$. Let E be the intersection of the two diagonals. Find the ratio

$$\frac{\overline{BE}}{\overline{DE}}.$$

15. In a triangle ABC the medians from vertices B and C are perpendicular. If $\overline{AB} = 19$ and $\overline{AC} = 22$, what is \overline{BC} ?

16. A square $ABCD$ is given. A circle with radius AB and center A is drawn. This circle intersects the perpendicular bisector of BC in two points, of which O is the closest to C . Find the value of $\angle AOC$ in degrees.

17. A line segment AB of length 10 is tangent to a circle of radius 10 at the point A . If we rotate the circle and the tangent line 60 degrees about the center of the circle, what is the area of the region swept out by AB ?

18. A frog and a grasshopper are 2 meters apart. Each second, the frog moves either 25 cm or 50 cm on the ground towards the grasshopper, and also the grasshopper jumps either 25 cm or 50 cm towards the frog. The frog will eat the grasshopper if the two reach each other on the ground. In how many ways can the grasshopper be eaten by the frog?

19. Solve the equation $x = \sqrt{a - \sqrt{a + x}}$, where $a > 0$ is a parameter.