DE EXAM SOLUTIONS
Texas A&M High School Math Contest
November 2016

1. Find the length of an edge of an equilateral triangle that has one corner at $(0, 0)$ and the other two on the graph of $xy = 1$. Simplify fully.

   Solution: The only way to get an acute angle in the triangle is to take the other two points on the same component of the graph, and they might as well be on the component where x and y are positive.

   The symmetry of the triangle required, and the symmetry of the curve, suggest that the points we're looking for will have the form $(x, 1/x)$ and $(1/x, x)$. If so, we will have to have $(x-1/x)^2+(1/x-x)^2 = x^2+1/x^2 = 1/x^2 + x^2$. Simplifying and multiplying through by x^2 leads to just one equation: $0 = x^4 - 4x^2 + 1$. This is a quadratic equation in the unknown x^2 and the solution is $x^2 = (4 \pm \sqrt{12})/2 = (2 \pm \sqrt{3})$. We can continue with the solution using either the plus or the minus here. Say $x^2 = 2 + \sqrt{3}$. Then $1/x^2 = 1/(2 + \sqrt{3}) = (2 - \sqrt{3})/(4 - 3) = 2 - \sqrt{3}$ so $x^2 + 1/x^2 = 4$. Thus the side length is 2.

2. In quotient and remainder division of one positive integer p by another, d, the quotient q is the largest integer q such that $qd \leq p$, and the remainder is $p - qd$. Thus, for instance, the remainder when dividing 22 by 5 is 2, and the quotient is 4.

   Find the smallest positive integer n so that when n is divided by 3, the remainder is 1, when n is divided by 5, the remainder is 2, when n is divided by 7, the remainder is 3, and when n is divided by 11, the remainder is 4.

   Solution: Whatever n is, it has to have the form $n = 3k + 1$ so that the remainder on division by 3 can be 1. Looking at successive values of $3k+1$, starting with $k = 0$, they go $1, 4, 7, 10, 13, 16, 19, 22$ and so on. The remainders from division by 5 go $1, 4, 2, 0, 3, 1, 4, 2, 0, 3$ and so on. To have a remainder of 2 when dividing by 5, as well as 1 mod 3, then, n must be in the arithmetic progression $(7, 22, 37, 52 \cdots)$, that is, $n = 7 + 15j$. To get a remainder of 3 when dividing by 7, note that the remainders in $7 + 15j$, when dividing by 7, go $(0, 1, 2, 3, 4, 5, 6, 0, \cdots)$ so that our new sequence is $n = 52 + 105l$. Finally, for remainders on dividing by 11, note that they're the same with $52 + 105l$ as they would be with $8 + 6l$. Those go $(8, 3, 9, 4)$ and we need look no further. Taking l to be not 0, or 1, or 2, but 3 has to work. Now $52 + 315 = 367$ so the answer is 367.

3. Multiplying $(1 + x + x^2 + \cdots + x^9)^{10}$ gives an expression

   $$c_0 + c_1x + c_2x^2 + \cdots + c_{90}x^{90} = 1 + 10x + 55x^2 + \cdots + 10x^{89} + x^{90}.$$

   Find $c_0 + c_1 + c_2 + \cdots + c_{90}$.

Solution: Setting $x = 1$ in the polynomial gives 10^{10}. In the expanded version, it gives $c_0 + c_1 + \cdots$ so the answer is ten billion, or 10^{10}, or 10000000000.

4. Let m be the integer whose binary (base 2) representation is 1010101. Find the binary representation of m^2.

   Solution: There's two ways to work this. Just write out a multiplication of 1010101 times itself and do it, but keep in mind that a 2 in position k must be replaced with a 1 in position $k + 1$ and so forth, or put m into familiar base 10 notation, square it, and convert back to binary.

   In the first approach, one gets 1110000111001. In the second, $m = 85$, $m^2 = 7225$, and the highest power of 2 that will fit in 7225 is $2^{12} = 4096$. What remains is 3129. Subtracting $2^{11} = 2048$ leaves 1081, subtracting $2^{10} = 1024$ leaves 57, and $57 = 32 + 16 + 8 + 1$. Thus, arranging the digits with the ones for higher powers of 2 on the left, we have 1110000111001 again.

5. Find the (unique) y so that $1/2 < y$ and

$$\frac{y^2}{\sqrt{1-y^2}} + \sqrt{1-y^2} = 2y.$$

   Solution: Multiply everything by $\sqrt{1-y^2}$ to get $1 = 2y\sqrt{1-y^2}$. Square that to get $1 = 4y^2(1-y^2)$ and then $4y^4 - 4y^2 + 1 = 0$. This factors as $(2y^2-1)^2$ so $y^2 = 1/2$. We wanted a positive value of y, so $y = 1/\sqrt{2}$.

6. Solve for x and y:

$$\begin{aligned} \log_2(x^3y^4) &= -2 \\ \log_4(x^5y^7) &= -1. \end{aligned}$$

   Solution: The log base 2 of a number is twice the log base 4. So if we write $X = \log_2 x$ and $Y = \log_2 y$ then $3X + 4Y = -2$ and $5X + 7Y = -2$. So then $2X + 3Y = 0$ and $X = -(3/2)Y$. Back to $3X + 4Y = -2$ yields $-Y/2 = -2$ so $Y = 4$ and $X = -6$. Thus $y = 16$ and $x = 1/64$.

7. Let P be a polynomial of degree 5. Given that $P(0) = 0$, and $P(1) = P(2) = P(3) = P(4) = P(5) = 1$, find $P(8)$.

   Solution: Let $Q = P - 1$. Then $Q(0) = -1$ while $Q(1)$ through $Q(5)$ are 0, so $Q = (1/120)(x-1)(x-2)(x-3)(x-4)(x-5)$. It follows that $Q(8) = 21$ and $P(8) = 22$. The answer is 22.

8. Let $\theta = \arctan 2 + \arctan 3$. Find $1/\sin^2\theta$ and simplify fully.

   Solution: Use the formula for $\sin(u+v)$. Right triangles with sides 1, 2, $\sqrt{5}$ for $\arctan 2$, and sides 1, 3, $\sqrt{10}$ for $\arctan 3$, yield values for the inputs to this formula, so

$$\sin\theta = \frac{2}{\sqrt{5}}\frac{1}{\sqrt{10}} + \frac{1}{\sqrt{5}}\frac{3}{\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

Consequently, the answer is 2.

9. Let $f(x) = |x| + |2x-1| - |3x-2|$. What is the area of the region inside the square $-1 \le x \le 1$, $-1 \le y \le 1$ and above the graph of $f(x)$? Write the answer as an improper fraction in lowest terms.

   Solution: The graph of $|ax+b|$ has constant slope except for a single angle, at the x where $ax+b=0$. This means that the graph of $f(x)$ consists of linked straight-line segments, with corners at $x=0$, $x=1/2$, and $x=2/3$. The slopes of the components of f are ∓ 1, ± 2, and ± 3, so that for $x<0$, the overall slope is 0, for $0<x<1/2$, the slope is 2, for $1/2<x<2/3$, the slope is 6, and for $x>2/3$, the slope is again 0. Now $f(0)=1-2=-1$, $f(1/2)=1/2+0-1/2=0$, and $f(2/3)=2/3+1/3-0=1$, so the overall graph goes in straight line segments from $(-1,-1)$ to $(0,-1)$ to $(1/2,0)$ to $(2/3,1)$ to $(1,1)$. The area above the first leg of the graph is 2, the area above the next leg is 3/4, and the next two legs contribute 1/12 and 0. This sums to $34/12 = 17/6$.

10. How many integer pairs (m,n) are there so that $0 \le n \le \sqrt{2}m$ and $m \le 10$?

    Solution: The answer is 84. There are 165 points inside the rectangle $[0,10]$ by $[0,14]$. Of these, 3 lie on the line $y = 7x/5$, half of the rest lie below, and half lie above it. Those below or on that line qualify, others do not. The slopes 7/5 and $\sqrt{2}$ are equivalent for purposes of this count because there are no integer points strictly between the lines $y = 7x/5$ and $y = 17/12$ (which has slope greater than $\sqrt{2}$) until after $x = 12$. And that, in turn, is true because the triangle with corners $(0,0)$, $(10,14)$, and $(12,17)$ has area 1, while if there were a lattice point in its interior, the three triangles that would result from connecting that interior point to $(0,0)$, $(10,14)$, and $(12,17)$ would have to have each an area of at least 1/2.

11. Find the sum of $5!/(a!b!c!)$ over all lists (a,b,c) of nonnegative integers so that $a+b+c=5$.

    Solution: the answer is 243. If you multiply out $(x+y+z)^5$, the coefficient of the term involving $x^a y^b z^c$ is $5!/(a!b!c!)$. Now set $x=y=z=1$.

12. Find a and b, with $b>0$, so that $(x-b)^2$ is a factor of $x^4 - ax + 1$.

    Solution: The answer is $a = 4(3)^{-3/4}$, $b = 3^{-1/4}$. For $(x-b)^2$ to be a factor, the quotient would have to be $x^2 + 2bx + 1/b^2$, so that the x^3 term of the product can be 0 and the constant term can be 1. Now in order to

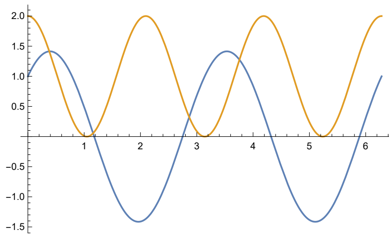

Figure 1: Intersecting Graphs

get the x^2 term in the product to be zero, b must be $\pm 3^{-1/4}$. This then forces the value for a. (The answer can also be given as

$$a = \frac{4 \cdot 3^{1/4}}{3}, \quad b = \frac{3^{3/4}}{3} = \frac{\sqrt{3\sqrt{3}}}{3}.$$

13. Find the exact value of $\tan \pi/8$ and simplify.

    Solution: The answer is $\sqrt{2} - 1$. Let A be the required answer. From trigonometry, $\tan 2x = 2 \tan x/(1 - \tan^2 x)$. Since $\tan \pi/4 = 1$, we have $1 = 2A/(1 - A^2)$. Thus $1 - A^2 = 2A$ and $A^2 + 2A - 1 = 0$. The quadratic formula now offers two possibilities: $A = \sqrt{2} - 1$ or $A = -\sqrt{2} - 1$. Since $A > 0$ because $0 < \pi/8 < \pi/2$, the answer must be $\sqrt{2} - 1$. (Some students may answer $(-2 + \sqrt{8})/2$ but this is not simplified sufficiently.)

14. How many real numbers x are there such that $\sin 2x + \cos 2x = 1 + \cos 3x$ and $0 < x < 2\pi$?

    Solution: There are four. Both expressions are sine waves. The LHS oscillates between $-\sqrt{2}$ and $\sqrt{2}$ completing two cycles over the given interval, while the RHS oscillates between 0 and 2, completing three cycles. A rough sketch of the two graphs gives good enough reason to think that there are exactly four, and more detailed calculations confirm it if one has the time. (see figure)

15. Let C be the unit cube with corners such that each coordinate is 0 or 1. (Thus, $(0, 0, 0)$ and $(1, 1, 1)$ are a pair of opposite corners.) Let H be the set of all points inside C and equally distant from those two corners. Let T be that part of the cube consisting of all points that are on some line segment joining $(1, 1, 1)$ to a point in H. Find the volume of T.

    Solution: The answer is $3/8$. H is a regular hexagon with each edge having length $1/\sqrt{2}$. (For instance, one edge goes from $(1/2, 0, 1)$ to $(1, 0, 1/2)$.) The area of any one of the six equilateral triangles that compose H is thus $\sqrt{3}/8$, so the area of H is $3\sqrt{3}/4$. The distance from H to $(0, 0, 0)$ is the distance from the center $(1/2, 1/2, 1/2)$ of H to $(0, 0, 0)$, and that's $\sqrt{3}/2$. The volume of a solid pyramid of base B and height h is $Bh/3$. Here, that works out to $3/8$.

16. Find integers A, B, C, D so that

$$\cos 3x = A\cos^3 x + B\cos^2 x + C\cos x + D.$$

Solution: $A = 4$ and $C = -3$; $B = D = 0$. Expand $\cos 3x = \cos x \cos 2x - \sin x \sin 2x$ and then use the double angle formulas to arrive at that answer.

17. Given that $x + y = A$ and $x^2 + y^2 = B$, express $x^4 + y^4$ in the form $PA^4 + QA^2B + RB^2$. That is, find values for P, Q, and R that ensure the identity holds whatever the values of x and y.

Solution: The answer is that $P = -1/2$, $Q = 1$, and $R = 1/2$. This may be checked by multiplying out the resulting expression for $PA^4 + QA^2B + RB^2$.

18. Find the least prime number that divides $10! + 1$.

Solution: There's a brute force solution, and there's a conceptual solution. The brute force solution is that $10! + 1 = 3628801$ and this is not divisible by any of 2, 3, 5, or 7. That leaves 11 as a candidate and sure enough the result is divisible by 11.

In general, for a conceptual solution, if p is prime then $(p-1)!$ is congruent to -1 mod p. That's because every number except $p-1$ and 1 has a multiplicative inverse different from itself mod p. (The exceptions 1 and $p-1$ are their own inverses). The product thus works out to a bunch of pairs of inverses, which multiply to 1 mod p, one stand-alone 1, and one stand-alone -1 mod p.

Here, that means that 11 divides $10! + 1$. The other prime factor (there's only one other) is $(10! + 1)/11$.

19. Two vertices of a triangle are $(0, 0)$ and $(1, 0)$. The third is taken at random from the line segment from $(-1, 1)$ to $(2, 1)$. What is the probability that the triangle entirely encloses the circle about $(1/2, 1/2)$ of radius $1/10$?

Solution: The answer is $1/6$. The line $y = 4x/3$ meets the circle $(x - 1/2)^2 + (y - 1/2)^2 = 1/100$ at a tangent at the point $(21/50, 28/50)$ and then hits the line $y = 1$ at $(3/4, 1)$. The line through $(1, 0)$ to $(1/4, 1)$ also is tangent to the circle. This means that the point on the line $y = 1$ must have an x coordinat between $1/4$ and $3/4$. Since the x coordinate of the random point can be anywhere from -1 to 2, the chance that it falls between $1/4$ and $3/4$ is $1/6$.