BEST STUDENT EXAM

Texas A&M High School Math Contest

October 21, 2017

Directions: Answers should be simplified, and if units are involved include them in your answer.

1. The lengths of the altitudes of a triangle are proportional to 15, 21 and 35. What is the largest internal angle of the triangle in degrees?

2. Simplify
$$\sqrt{\sqrt{(100)(102)(104)(106)+16}+5}.$$

3. Joshua randomly picks a positive perfect square less than 2017, Jay randomly picks a positive perfect cube less than 2017, and Jonathan randomly picks a positive perfect sixth power less than 2017. What is the probability that all three picked the same number?

4. Consider an equilateral triangle with side length $2a$. We make folds along the three line segments connecting the midpoints of its sides, until the vertices of the triangle coincide. What is the volume of the resulting tetrahedron?

5. Define a sequence $(x_n)_{n\geq 1}$ recursively by $x_1 = 0,\ \ x_n = \sqrt{2 + x_{n-1}}$ for $n \geq 2$. What is $\arccos\left(\frac{x_5}{2}\right)$ in radians?

6. Determine the remainder upon dividing $6^{2017} + 8^{2017}$ by 49.

7. Let
$$P_n = \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \cdots \times \frac{n^3-1}{n^3+1}.$$
Find $\lim_{n\to\infty} P_n$.

8. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^{2017} x}$.

9. For a sequence $A = (a_0, a_1, a_2, \dots)$ define a new sequence $\Delta(A) = (a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots)$. Suppose that $\Delta(\Delta(A)) = (1, 1, 1, \dots)$, and $a_{20} = a_{17} = 0$. Find a_0.

10. Find the height of the right circular cone of minimum volume which can be circumscribed about a sphere of radius R.

11. Find the greatest integer preceding $\displaystyle\sum_{n=1}^{10{,}000} \frac{1}{\sqrt{n}}$.

12. In an isosceles triangle $\triangle ABC$ $(AB = AC)$, the angle bisector of $\angle ACB$ divides the triangle $\triangle ABC$ into two other isosceles triangles. What is the ratio $\dfrac{|BC|}{|AB|}$?

13. Simplify the value of

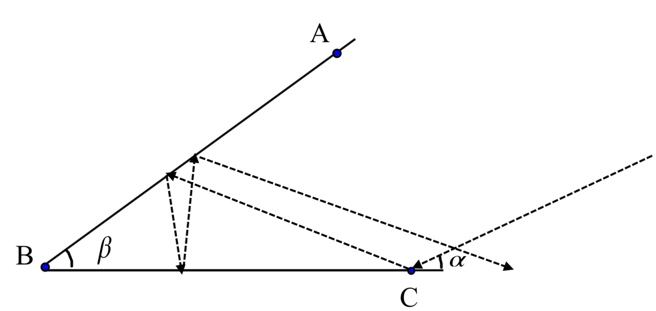
$$\frac{2018^4 + 4 \times 2017^4}{2017^2 + 4035^2} - \frac{2017^4 + 4 \times 2016^4}{2016^2 + 4033^2}.$$

14. How many pairs of positive integers x, y exist such that $x < y$ and $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{200}$?

15. Given that it converges, evaluate

$$\int_0^1 \int_0^1 \sum_{k=0}^{\infty} x^{(y+k)^2}\, dx\, dy.$$

16. A billiard ball (of infinitesimal diameter) strikes ray $\overrightarrow{BC}$ at point C, with angle of incidence $\alpha = 2.5°$. The billiard ball continues its path, bouncing off line segments $\overline{AB}$ and $\overline{BC}$, which are making an angle $\beta = 17°$, according to the rule "angle of incidence equals angle of reflection." If $AB = BC$, determine the number of times the ball will bounce off the two line segments (including the first bounce, at C).



17. Let $x_1, x_2, \ldots, x_7$ be the roots of the the polynomial $P(x) = \sum_{k=0}^{7} a_k x^k$, where

$$a_0 = a_4 = \mathbf{2}, \quad a_1 = a_5 = \mathbf{0}, \quad a_2 = a_6 = \mathbf{1}, \quad a_3 = a_7 = \mathbf{7}.$$

Find

$$\frac{1}{1-x_1} + \frac{1}{1-x_2} + \cdots + \frac{1}{1-x_7}.$$

18. A fair coin is tossed 9 times. What is the probability that no two consecutive heads appear?