BEST STUDENT EXAM SOLUTIONS
Texas A&M High School Math Contest
October 21, 2017

Directions: Answers should be simplified, and if units are involved include them in your answer.

1. The lengths of the altitudes of a triangle are proportional to 15, 21 and 35. What is the largest internal angle of the triangle in degrees?

   Answer. 120 degrees.
   The sides of the triangle are proportional to 1/15, 1/21, and 1/35, or in other words, to 7, 5, 3. Let the largest angle be θ, which is opposite to the side that is proportinal to 7. By the Law of Cosines, we will have

   $$7^2 = 5^2 + 3^2 - 2(5)(3)\cos\theta.$$

   This leads to $\cos\theta = -1/2$, hence $\theta = 120°$.

2. Simplify

   $$\sqrt{\sqrt{(100)(102)(104)(106)+16}+5}.$$

   Answer. 103.
   Use the identity

   $$(a-3)(a-1)(a+1)(a-3)+16 = (a^2-9)(a^2-1)+16 = a^4-10a^2+9+16 = (a^2-5)^2$$

   for $a = 103$.

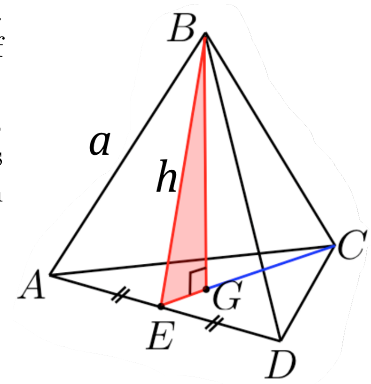
3. Joshua randomly picks a positive perfect square less than 2017, Jay randomly picks a positive perfect cube less than 2017, and Jonathan randomly picks a positive perfect sixth power less than 2017. What is the probability that all three picked the same number?

   Answer. $1/(44 \times 12) = 1/528$.
   Since all perfect sixth powers are also perfect squares and perfect cubes as well, the value we need to find is simply the probability that both Joshua and Jay picked the same number that Jonathan picked. There are 44 different perfect squares that Joshua could have picked (since $44^2 = 1936 < 2017 < 2025 = 45^2$), and there are 12 different perfect cubes that Jay could have picked (since $12^3 = 1728 < 2017 < 2197 = 13^3$). Thus the probability that Joshua picked the same number as Jonathan is 1/44, and the probability that Jay picked the same number as Jonathan is 1/12. The answer is thus $1/(44 \times 12) = 1/528$.

4. Consider an equilateral triangle with side length $2a$. We make folds along the three line segments connecting the midpoints of its sides, until the vertices of the triangle coincide. What is the volume of the resulting tetrahedron?

   Answer. $a^3\sqrt{2}/12$.

The resulting tetrahedron is obviously regular. The base of the tetrahedron is an equilateral triangle of side length a, so the area of the base is $A_b = a^2\sqrt{3}/4$. Let's find the height H of the tetrahedron. In the figure, note that the foot G of the altitude BG is, indeed, the centroid of the base triangle (e.g, by symmetry). Let $h = |BE| = |CE|$ be the height of the equilateral faces of the tetrahedron, so $h = a\sin(\pi/3) = a\sqrt{3}/2$. On the other hand, since $\triangle ADC$ is equilateral, its centroid G coincides with its orthocenter, so $|GE| = h/3$. Applying Pythagorean theorem for $\triangle BGE$ results in



$$H = |BG| = \sqrt{|BE|^2 - |GE|^2} = \sqrt{h^2 - \left(\frac{h}{3}\right)^2} = \frac{2\sqrt{2}}{3}h = a\frac{\sqrt{6}}{3}.$$

Consequently, $V = A_b \cdot H/3 = a^3\sqrt{2}/12$.

5. Define a sequence $(x_n)_{n\geq 1}$ recursively by $x_1 = 0,\ \ x_n = \sqrt{2 + x_{n-1}}$ for $n \geq 2$. What is $\arccos\left(\frac{x_5}{2}\right)$ in radians?

Answer. $\pi/32$ radians.
Note that, using the half-angle formula

$$1 + \cos(\alpha) = 2\cos^2(\alpha/2),$$

we can write

$$x_3 = \sqrt{2+\sqrt{2}} = \sqrt{2 + 2\cos(\pi/4)} = \sqrt{2(1+\cos(\pi/4))} = \sqrt{4\cos^2(\pi/8)} = 2\cos(\pi/8).$$

Similarly,

$$x_4 = \sqrt{2 + x_3} = \sqrt{2 + 2\cos(\pi/8)} = \sqrt{4\cos^2(\pi/16)} = 2\cos(\pi/16),$$

and

$$x_5 = \sqrt{2 + x_4} = \sqrt{2 + 2\cos(\pi/16)} = \sqrt{4\cos^2(\pi/32)} = 2\cos(\pi/32).$$

6. Determine the remainder upon dividing $6^{2017} + 8^{2017}$ by 49.

Answer. 14.
We have $7 - 1 = 6, 7 + 1 = 8$, and $7^2 = 49$. By Binomial Theorem, we have

$$(7-1)^{2017} = 7(2017) - 1, \quad (7+1)^{2017} = 7(2017) + 1 \pmod{49}.$$

Also, $2017 = 41(49) + 8$, so $6^{2017} + 8^{2017} = 7(8) - 1 + 7(8) + 1 = 14 \pmod{49}$.

Another method uses the Euler totient function $\phi(n)$. We know $\phi(49) = 49 - 7 = 42$. Therefore, by Euler's theorem, we have

$$6^{42} = 1, \quad 8^{42} = 1 \pmod{49}$$

because gcd(6,49)=gcd(8,49)=1. On the other hand, $2017 = 48(42) + 1$,

$$6^{2017} + 8^{2017} = (6^{42})^{48} \times 6 + (7^{42})^{48} \times 7 = 6 + 8 = 14 \pmod{49}.$$

7. Let

$$P_n = \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \cdots \times \frac{n^3-1}{n^3+1}.$$

Find $\lim_{n\to\infty} P_n$.

Answer. 2/3.
Note that

$$\frac{n^3-1}{n^3+1} = \frac{(n-1)[n(n+1)+1]}{(n+1)[n(n-1)+1]}.$$

Let's re-formulate P_n:

$$P_n = \frac{(1)[2(3)+1]}{(3)[2(1)+1]} \times \frac{(2)[3(4)+1]}{(4)[3(2)+1]} \times \frac{(3)[5(6)+1]}{(5)[4(3)+1]} \times \cdots$$
$$\cdots \times \frac{(n-3)[(n-2)(n+1)+1]}{(n-1)[(n-2)(n-3)+1]} \times \frac{(n-2)[(n-1)(n)+1]}{(n)[(n-1)(n-2)+1]} \times \frac{(n-1)[n(n+1)+1]}{(n+1)[n(n-1)+1]}.$$

After cancellations, we have

$$P_n = \frac{1(2)}{n(n+1)} \times \frac{n(n+1)+1}{2(1)+1} \longrightarrow \frac{2}{3}, \text{ as } n \to \infty.$$

8. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^{2017} x}$.

Answer. $\pi/4$.
Let's call the integral I. So we have

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^{2017} x}{\sin^{2017} x + \cos^{2017} x}\, dx.$$

Substitute $x \mapsto \pi/2 - x$ to get

$$I = \int_{\frac{\pi}{2}}^0 \frac{\sin^{2017} x}{cos^{2017} x + \sin^{2017} x}\, (-dx) = \int_0^{\frac{\pi}{2}} \frac{\sin^{2017} x}{\sin^{2017} x + \cos^{2017} x}\, dx.$$

Adding the two integrals leads to

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^{2017} x + \sin^{2017} x}{\sin^{2017} x + \cos^{2017} x}\, dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$

9. For a sequence $A = (a_0, a_1, a_2, \ldots)$ define a new sequence $\Delta(A) = (a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots)$. Suppose that $\Delta(\Delta(A)) = (1, 1, 1, \ldots)$, and $a_{20} = a_{17} = 0$. Find a_0.

Answer. 170.
Let us find the sequence "backward." The sequence $\Delta(A)$ is of the form $(b, b+1, b+2, b+3, \ldots)$. Consequently, the sequence A is of the form $(a_0, a_0 + b, a_0 + 2b + 1, a_0 + 3b + 3, \ldots)$, and we get the formula

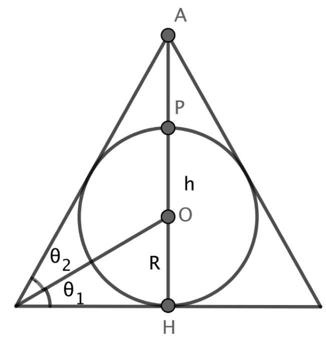
$$a_n = a_0 + nb + \frac{n(n-1)}{2}$$

through summing up the difference terms $a_k - a_{k-1} = b + k - 1$ from $k = 1$ to $k = n$. Then $a_0 = 170$ and $b = -18$ are obtained by solving the system $0 = a_{20} = a_0 + 20b + 190$, and $0 = a_{17} = a_0 + 17b + 136$.

10. Find the height of the right circular cone of minimum volume which can be circumscribed about a sphere of radius R.

Answer. $4R$.

The figure shows a cross section of the two shapes through the axis of the cone. It is clear that since O is the incenter of the triangle, we have $\theta_1 = \theta_2$, so let us call them θ. Let r be the radius of the base and h be the height of the cone. Therefore, $r = R\cot\theta$ and $h = |AH| = R\cot\theta\tan 2\theta$. To minimize $V = \pi/3\, r^2 h$, we must minimize



$$\cot^3\theta\,\tan 2\theta = \cot^3\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \frac{2}{\tan^2\theta - \tan^4\theta},$$

or equivalently, we must maximize $\tan^2\theta - \tan^4\theta$. Upon taking the derivative of the latter function, we arrive at

$$(2\tan\theta - 4\tan^3\theta)\sec^2\theta = 0,$$

which gives $\tan^2\theta = 1/2$, hence

$$h = R\cot\theta\tan 2\theta = R\cot\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \frac{2R}{1-1/2} = 4R.$$

11. Find the greatest integer preceding $\displaystyle\sum_{n=1}^{10{,}000} \frac{1}{\sqrt{n}}$.

Answer. 198.
Let $S = \sum_{n=1}^{10{,}000} \frac{1}{\sqrt{n}}$. We use the inequalities $\sqrt{n} + \sqrt{n-1} < 2\sqrt{n} < \sqrt{n+1} + \sqrt{n}$ and rationalize the reciprocals to get

$$2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1}).$$

Note that the first term of S is 1, so adding up over the right sides of above inequalities for $n = 2, \ldots, 10000$ and telescoping results in

$$S - 1 < 2(\sqrt{10{,}000} - 1),$$

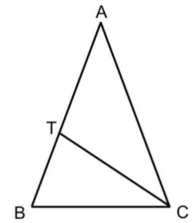
which implies that $S < 199$, whereas adding up over the left sides for $n = 1, \ldots, 10000$ and telescoping results in

$$2(\sqrt{10{,}001} - 1) < S,$$

which implies that $S > 198$ because $198/2 + 1 = 100 < \sqrt{10{,}001}$. In summary, $198 < S < 199$, so the greatest integer preceding S is 198.

12. In an isosceles triangle $\triangle ABC$ ($AB = AC$), the angle bisector of $\angle ACB$ divides the triangle $\triangle ABC$ into two other isosceles triangles. What is the ratio $\dfrac{|BC|}{|AB|}$?

Answer. $(\sqrt{5} - 1)/2$.



Let T be the point where the side AB and the angle bisector of $\angle ACB$ intersect. By angle bisector theorem. we have

$$\frac{|AC|}{|BC|} = \frac{|AT|}{|BT|},$$

Note that by construction $|AT| < |AC|$ and $\angle TCB < \angle TBC$ so that $|BT| < |BC|$. Consequently, the assumption of the problem is only possible if $|AT| = |BC| = |CT|$. Therefore,

$$\frac{|BC| + |BT|}{|BC|} = \frac{|BC|}{|BT|}.$$

In particular,

$$\frac{|BC|}{|AB|} = \frac{|BC|}{|BC| + |BT|} = \frac{|BT|}{|BC|}.$$

Letting x be the common ratio as above, the later equality implies $x = 1/(1+x)$, so $x = (\sqrt{5}-1)/2$.

13. Simplify the value of

$$\frac{2018^4 + 4 \times 2017^4}{2017^2 + 4035^2} - \frac{2017^4 + 4 \times 2016^4}{2016^2 + 4033^2}.$$

Answer. 4,033.

We know by completing the square that $a^4 + 4b^4 = (a^4 + 4a^2b^2 + b^4) - 4a^2b^2$, therefore

$$a^4 + 4b^4 = (a^2 + 2b^2)^2 - 4a^2b^2 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2).$$

In particular, for $a = b + 1$ we have

$$(b+1)^4 + 4b^4 = (1 + b^2)[b^2 + (2b+1)^2].$$

As a consequence, we have

$$\frac{(b+1)^4 + 4b^4}{[b^2 + (2b+1)^2]} = 1 + b^2.$$

Note that the left side of the above equality is precisely of the form of each of the two fractions given in the problem. So the given number simplifies to $(1 + 2017^2) - (1 + 2016^2) = (2017 - 2016)(2017 + 2016) = 4033$.

14. How many pairs of positive integers x, y exist such that $x < y$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{200}$?

Answer. 17.
By positivity of x, y, it is clear that $200 < x < y$. Parametrizing $x = 200 + k$ with a natural number k results in $y = 200(200+k)/k = 40,000/k + 200$, where $k < 200$ must divide $40,000$. Since $40,000 = 2^6 \times 5^4$, all divisors of $40,000$ are of the form $k = 2^m \times 5^n \quad (0 \le m \le 6, 0 \le n \le 4)$, so those that are less than 200 correspond to the following 17 pairs of exponents: $(m, 0)$ for $0 \le m \le 6$, $(m, 1)$ for $0 \le m \le 5$, $(m, 2)$ for $0 \le m \le 2$, and $(m, 3)$ for $m = 0$.

15. Given that it converges, evaluate

$$\int_0^1 \int_0^1 \sum_{k=0}^{\infty} x^{(y+k)^2}\ dx\ dy.$$

Answer. $\pi/2$.
We are given that the integral converges, and since the integrand is positive everywhere, Fubini's theorem dictates that it must converge absolutely. Thus we are free to rearrange the integrals. We move the integral with respect to x inside the sum and evaluate it first.

$$\int_0^1 x^{(y+k)^2}\ dx = \frac{1}{(y+k)^2+1}$$

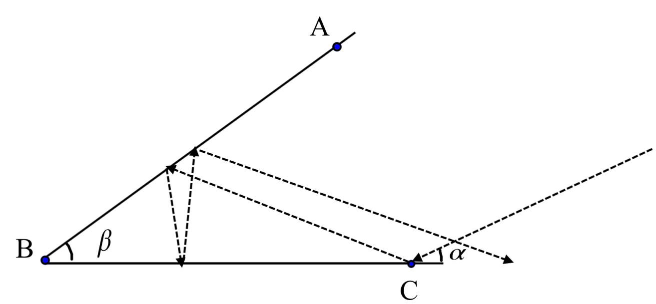
Now we move the integral with respect to y inside the sum and evaluate it as well.

$$\int_0^1 \frac{1}{(y+k)^2+1}\ dy = \arctan(y+k)\Big|_{y=0}^{1} = \arctan(k+1) - \arctan(k).$$

Finally we evaluate the sum, which telescopes nicely.

$$\sum_{k=0}^{\infty}(\arctan(k+1) - \arctan(k)) = \lim_{N\to\infty}\sum_{k=0}^{N}(\arctan(k+1) - \arctan(k)) = \lim_{N\to\infty}(\arctan(N+1) - \arctan(0)) = \lim_{N\to\infty}\arctan(N) = \pi/2.$$

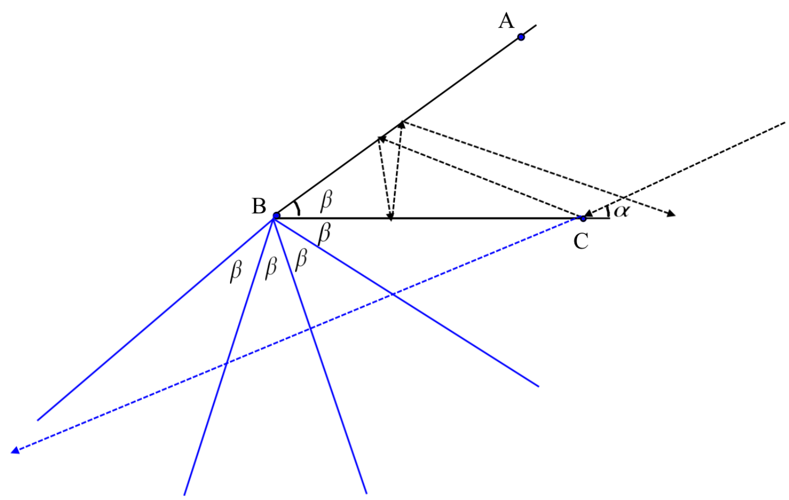
16. A billiard ball (of infinitesimal diameter) strikes ray $\overrightarrow{BC}$ at point C, with angle of incidence $\alpha = 2.5°$. The billiard ball continues its path, bouncing off line segments $\overline{AB}$ and $\overline{BC}$, which are making an angle $\beta = 17°$, according to the rule "angle of incidence equals angle of reflection." If $AB = BC$, determine the number of times the ball will bounce off the two line segments (including the first bounce, at C).



Answer. 11.
By the exterior angle theorem, every time the billiard ball bounces, the angle of incidence increases by β. When the ball begins to bounce backwards, we can interpret this as an obtuse angle of incidence. When the angle exceeds $180 - \beta$, the ball will not hit the walls anymore. If the ball bounds k times, including the first bounce at C, then k is the minimal positive integer such that $\alpha + (k-1)\beta \geq 180 - \beta$, or in other words, $k\beta \geq 180 - \alpha$. This means that $k \geq (180-\alpha)/\beta$, hence

$$k = \left\lceil \frac{180-\alpha}{\beta} \right\rceil,$$

where $\lceil x \rceil$ means the least integer greater than or equal to x. For this problem, $\lceil (180-\alpha)/\beta \rceil = \lceil 177.5/17 \rceil = 11$.



17. Let $x_1, x_2, \ldots, x_7$ be the roots of the the polynomial $P(x) = \sum_{k=0}^{7} a_k x^k$, where

$$a_0 = a_4 = \mathbf{2}, \quad a_1 = a_5 = \mathbf{0}, \quad a_2 = a_6 = \mathbf{1}, \quad a_3 = a_7 = \mathbf{7}.$$

Find

$$\frac{1}{1-x_1} + \frac{1}{1-x_2} + \cdots + \frac{1}{1-x_7}.$$

Answer. 43/10.
A change of variable of the form

$$y = \frac{1}{1-x}$$

will be convenient for this problem. This gives $x = (y-1)/y$, which converts the polynomial equation

$$2 + x^2 + 7x^3 + 2x^4 + x^6 + 7x^7 = 0$$

into

$$2 + \left(\frac{y-1}{y}\right)^2 + 7\left(\frac{y-1}{y}\right)^3 + 2\left(\frac{y-1}{y}\right)^4 + \left(\frac{y-1}{y}\right)^6 + 7\left(\frac{y-1}{y}\right)^7 = 0,$$

which, after multiplying by y^7 reads

$$\sum_{k=0}^{7} b_k y^k = 2y^7 + (y-1)^2 y^5 + 7(y-1)^3 y^4 + 2(y-1)^4 y^3 + (y-1)^6 y + 7(y-1)^7 = 0.$$

Now, all we need is the sum of the 7 roots of the latter polynomial equation, which is known to be of the form $-b_6/b_7$. Using the binomial expansion

$$(y-1)^k = y^k - ky^{k-1} + \text{ lower order terms}$$

we have

$$b_6 = -(2 + 3 \times 7 + 4 \times 2 + 6 + 7 \times 7) = -86,$$

and

$$b_7 = 2 + 1 + 7 + 2 + 1 + 7 = 20.$$

Therefore, the desired sum is $86/20 = 43/10$.

18. A fair coin is tossed 9 times. What is the probability that no two consecutive heads appear?

Answer. 89/512.
There is a total of $2^9 = 512$ outcomes. We will find the number of favorable outcomes recursively. Namely, assume that the coin is tossed n times. and let f_n be the number of favorable outcomes. What we are trying to find is the number of favorable outcomes f_9, and the probability of the desired event is $f_9/512$.

If the first trial is T, then the second one can be either T or H. However, if the first trial is H, then the second one must be T. In summary, the sequence either starts with T, or with HT and the rest of them can be any favorable sequence. Consequently, $f_n = f_{n-1} + f_{n-2}$, so the sequence (s_n) is a Fibonacci sequence with $f_1 = 2$ (i.e., H, T), and $f_2 = 3$ (i.e., TT, TH, HT). So we have

$$f_3 = 5, f_4 = 8, f_5 = 13, f_6 = 21, f_7 = 34, f_8 = 55, f_9 = 89.$$