Solutions 2018 AB Exam

Texas A&M High School Math Contest
October 20

1. Find the rational number in the open interval $(\frac{1}{3}, \frac{1}{2})$ that has the smallest positive denominator.

   ANSWER: $\frac{2}{5}$

   Solution: With a and b positive integers we want

   $$\frac{1}{3} < \frac{a}{b} < \frac{1}{2} \text{ i.e.,}$$
   $$2a < b < 3a.$$

   We seek the smallest positive integer a such that the interval $(2a, 3a)$ contains an integer. This is $a = 2$ giving $(4, 6)$ and $b = 5$. Our candidate is $\frac{a}{b} = \frac{2}{5}$. Five is minimal since $(\frac{1}{3}, \frac{1}{2})$ contains no fraction of the form $\frac{a}{1}, \frac{a}{2}, \frac{a}{3}$ or $\frac{a}{4}$.

2. Let L be the line with equation $ax + by = c$ where $abc \neq 0$. Let M be the reflection of L across the y-axis. Let N be the reflection of L across the x-axis. Which of the following must be true about M and N?

   (A) The x-intercepts are equal.
   (B) The y-intercepts are equal.
   (C) The slopes are equal.
   (D) The slopes are reciprocals.
   (E) The slopes are negative reciprocals.

   ANSWER: C

   Solution: The intercepts of L are $(\frac{c}{a}, 0), (0, \frac{c}{b})$.
   The intercepts of M are $(-\frac{c}{a}, 0), (0, \frac{c}{b})$.
   The intercepts of N are $(\frac{c}{a}, 0), (0, -\frac{c}{b})$.
   The slope of M is $m_M = \frac{\frac{c}{b}}{\frac{c}{a}} = \frac{a}{b}$.
   The slope of N is $m_N = \frac{-\frac{c}{b}}{0 - \frac{c}{a}} = \frac{a}{b}$.

3. The sum of two natural numbers a and b is equal to 153. What is the largest possible value of their greatest common divisor, $gcd(a, b)$?

ANSWER: 51

Solution. If k is the divisor of both a and b, then $a = kc, b = kd$ for some natural numbers c, d and so $153 = a+b = k(c+d)$. On the other hand, $153 = 3 \cdot 3 \cdot 17$, so $k = 1, 3, 9, 17$, or 51. Thus, the largest possible value of $gcd(a, b)$ is 51 that works when $c = 1$, $d = 2$ $(a = 51, b = 102)$ or $c = 2$, $d = 1$ $(a = 102, b = 51)$.

4. Find all ordered pairs (a, b) such that $a + b = 16$ and $\frac{1}{a} + \frac{1}{b} = \frac{4}{7}$.

ANSWER: $(2, 14), (14, 2)$

Solution: $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{16}{ab} = \frac{4}{7}$

So $ab = 28$ and $b = \frac{28}{a}$. We have

$$16 = a + b = a + \frac{28}{a}$$
$$a^2 - 16a + 28 = 0$$
$$(a - 14)(a - 2) = 0$$

$a = 2$ means $b = 14$.
$a = 14$ means $b = 2$.

5. A certain medication has an 80% success rate of curing people who have a specific illness. If three people with the illness are selected at random and take the medicine, what is the probability that exactly two of the three people will be cured?

ANSWER: $\frac{48}{125} = 0.384$

Solution: The two "cured" people may be chosen in $\binom{3}{2} = 3$ different ways. The probability in each case is $\frac{8}{10}\frac{8}{10}\frac{2}{10} = \frac{4}{5}\frac{4}{5}\frac{1}{5} = \frac{16}{125}$. There are 3 different ways giving $(3)(\frac{16}{125}) = \frac{48}{125} = 0.384$.

6. If $ax + 3y = 5$ and $2x + by = 3$ represent the same line, find the value of $a + b$.

ANSWER: $\frac{77}{15}$

Solution: The y-intercept of $ax + 3y = 5$ is $(0, \frac{5}{3})$. So $2(0) + b(\frac{5}{3}) = 3$ and $b = \frac{9}{5}$.
The x-intercept of $2x + by = 3$ is $(\frac{3}{2}, 0)$. So $a(\frac{3}{2}) + 3(0) = 5$ and $a = \frac{10}{3}$.
$a + b = \frac{10}{3} + \frac{9}{5} = \frac{50}{15} + \frac{27}{15} = \frac{77}{15}$.

7. Let $S = \{1, 2, 3, ..., 11\}$. A subset of S of size 3 is said to be "special" if it contains at most one odd integer. If a subset of S of size 3 is chosen at random, what is the probability it is special?

ANSWER: $\frac{14}{33}$

Solution: The number of subsets of S if size 3 that contain no odd integers is

$$\binom{5}{3} = 10.$$

The number of size 3 subsets that contain exactly one odd integer is

$$\binom{6}{1}\binom{5}{2} = 6 \cdot 10 = 60.$$

The probability of choosing a special subset is

$$\frac{10+60}{\binom{11}{3}} = \frac{70}{165} = \frac{14}{33}.$$

8. When the integer $D > 1$ is divided into each of the numbers 1059, 1417, 2312 the same remainder R is obtained. Find D.

ANSWER: 179

Solution: $1059 = Q_1 D + R$

$$1417 = Q_2 D + R$$
$$2312 = Q_3 D + R$$

D is a common divisor of $1417 - 1059 = 358$ and $2312 - 1417 = 895$. Also D divides $895 - 358 = 537$ and D divides $537 - 358 = 179$.

Since 179 is prime, $D = 179$.

9. Determine all linear functions $f(x) = ax + b$ such that $f(x) - f^{-1}(x) = 44$ for all x. Note: the symbol f^{-1} denotes the inverse function of f, and not its reciprocal.

ANSWER: $f(x) = x + 22$

Solution: $f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}$

$$44 = f(x) - f^{-1}(x) = ax + b - \tfrac{1}{a}x + \tfrac{b}{a} \text{ for all } x$$

means

$$a - \frac{1}{a} = 0 \text{ and } b + \frac{b}{a} = 44.$$

So $a = 1$ or -1.
If $a = 1$ then $2b = 44$ and $b = 22$.
If $a = -1$ then $b + \frac{b}{a} = 44$ has no solution.
So $a = 1, b = 22$ gives

$$f(x) = x + 22.$$

10. For some real values of p, two of the roots of $x^3 + px^2 + 12x - 9 = 0$ have a sum of 4. Find the third root.

ANSWER: $\frac{3}{2}$

Solution: $x^3 + px^2 + 12x - 9 = (x - \alpha)(x - \beta)(x - \delta)$ where

$$\begin{aligned} \alpha + \beta &= 4 \\ \alpha\beta\delta &= 9 \\ \alpha\beta + \alpha\delta + \beta\delta &= 12 \end{aligned}$$

From second equation we obtain $\alpha\beta = 9/\delta$. Substituting this to the third equation, we get $9/\delta + \delta(\alpha + \beta) = 12$, and since $\alpha + \beta = 4$, we have $9/\delta + 4\delta = 12$ or $4\delta^2 - 12\delta + 9 = 0$. Finally, we obtain $(2\delta - 3)^2 = 0$ which implies $\delta = 3/2$.

11. The equation $(x^2 - x + 1)(x^2 - x + 2) = 12$ has two real solutions. Find their product.

ANSWER: -2

Solution:

$$\begin{aligned} (x^2 - x + 1)(x^2 - x + 2) &= 12 \\ (x^2 - x + 1)[(x^2 - x + 1) + 1] &= 12 \\ (x^2 - x + 1)^2 + (x^2 - x + 1) &= 12 \\ (x^2 - x + 1)^2 + (x^2 - x + 1) - 12 &= 0 \\ [(x^2 - x + 1) + 4][(x^2 - x + 1) - 3] &= 0 \\ (x^2 - x + 5)(x^2 - x - 2) &= 0 \end{aligned}$$

$$\Rightarrow \quad x^2 - x + 5 = 0 \quad \text{(no real solutions)}$$

$$x^2 - x - 2 = 0 \quad \text{(solutions } x = 2, -1\text{)}$$

So

$$(2)(-1) = \underline{-2}.$$

12. Hasse traveled 1 hour longer and 2 miles farther than Daisy but averaged 3 mph slower. If the sum of their times was 4 hours, what was the sum in miles of the distances they traveled?

ANSWER: $\frac{61}{2}$ or 30.5

Solution:

Hasse: $d_1 = r_1 t_1$, where d_1 is distance traveled
$r_1 =$ average rate, $t_1 =$ time elapsed.

Daisy: $d_2 = r_2 t_2$

We are given $t_2 + 1 = t_1$

$$t_1 + t_2 = 4$$
$$r_1 + 3 = r_2$$
$$d_2 + 2 = d_1$$

So easily $t_2 = \frac{3}{2}$ and $t_1 = \frac{5}{2}$.

$$d_2 - d_1 = r_2 t_2 - r_1 t_1$$
$$-2 = (r_1 + 3)\tfrac{3}{2} - r_1\tfrac{5}{2} = -r_1 + \tfrac{9}{2}$$

So $r_1 = 2 + \frac{9}{2} = \frac{13}{2}$ and $r_2 = r_1 + 3 = \frac{13}{2} + \frac{6}{2} = \frac{19}{2}$

$d_1 + d_2 = \frac{13}{2}\frac{5}{2} + \frac{19}{2}\frac{3}{2} = \frac{65+57}{4} = \frac{122}{4} = \frac{61}{2} = 30.5$

13. Two numbers when written in base a are 32 and 24. These same two numbers written in base b give 43 and 33 respectively. Find the sum of the two numbers in base 10.

ANSWER: 41

Solution: Let x and y be the two numbers.

$$x = 3a + 2 = 4b + 3$$
$$y = 2a + 4 = 3b + 3$$

$$3a - 4b = 1$$
$$2a - 3b = -1$$

Solving gives $a = 7, b = 5$.

So $x = 23$ and $y = 18$

$x + y = \underline{41}$.

14. The average age of the people in a group of men and women is 31 years. The average age of the men is 35 and the average age of the women is 25. What is the ratio of the number of men to the number of women? Write your answer as a simple fraction.

ANSWER: $\frac{3}{2}$

Let $M =$ the number of men, and $F =$ the number of women. Then we have:

$$\begin{aligned}\frac{35M+25F}{M+F} &= 31 \implies \frac{35\frac{M}{F}+25}{\frac{M}{F}+1} = 31 \implies 4\frac{M}{F} = 6\\ \frac{M}{F} &= \frac{3}{2}\end{aligned}$$

15. The point $(1, 2)$ is on the line $y = 2x$. Find the x-coordinate of each of the two points on $y = 2x$ that are 10 units from $(1, 2)$.

ANSWER: $2\sqrt{5}+1, -2\sqrt{5}+1$

Solution: $(x-1)^2 + (y-2)^2 = 10^2 = 100$

and $y = 2x$. So

$$\begin{aligned}(x-1)^2 + (2x-2)^2 &= 100\\ 5(x-1)^2 &= 100\\ (x-1)^2 &= 20\\ x-1 &= \pm\sqrt{20} = \pm 2\sqrt{5}\\ x &= 2\sqrt{5}+1 \text{ or } x = -2\sqrt{5}+1\end{aligned}$$

16. For exactly two real values m_1, and m_2 of m the line $y = mx + 3$ intersects the parabola $y = x^2 + 2x + 7$ at exactly one point. Find $m_1 + m_2$.

ANSWER: 4

Solution: $mx + 3 = x^2 + 2x + 7$

$x^2 + (2-m)x + 4 = 0.$

This quadratic must have one (repeated) root, so

$$\begin{aligned}(2-m)^2 - 16 &= 0\\ m^2 - 4m - 12 &= 0\end{aligned}$$

So, $m_1 + m_2 = 4$.

17. Suppose f and g are functions such that $f(x) = 2x + 1$ and $g(f(x)) = 4x^2 + 1$, find $g(x)$.

ANSWER: $g(x) = x^2 - 2x + 2$

Solution: We have $g(f(x)) = 4x^2 + 1 = (2x + 1)^2 - 4x = (2x + 1)^2 - 2(2x + 1) + 2 = (2x + 1)^2 - 2(2x + 1) + 2$. Thus, $g(x) = x^2 - 2x + 2$.

Check: $$\begin{aligned} g(f(x)) &= g(2x+1) = (2x+1)^2 - 2(2x+1) + 2 \\ &= 4x^2 + 4x + 1 - 4x - 2 + 2 \\ &= 4x^2 + 1 \end{aligned}$$

18. A stock loses 10% of its value on Monday. On Tuesday it loses 20% of the value it had at the end of the day on Monday. What is the overall percent loss in value from the beginning of Monday to the end of Tuesday?

ANSWER: 28%

Solution: Let V be its value at the beginning. At the end of Monday its value is

$$V - \tfrac{10}{100}V.$$

At the end of Tuesday the stock value is

$$\begin{aligned} &(V - \tfrac{10}{100}V) - \tfrac{20}{100}(V - \tfrac{10}{100}V) = \\ &V - \tfrac{10}{100}V - \tfrac{20}{100}V + \tfrac{2}{100}V = \\ &V - \tfrac{28}{100}V \end{aligned}$$

19. Solve for x the following equation

$$\sqrt{x} + \sqrt{x - 1} = 2.$$

ANSWER: $\frac{25}{16}$

Solution: Square both sides:

$$x + 2\sqrt{x}\sqrt{x-1} + x - 1 = 4$$
$$2\sqrt{x}\sqrt{x-1} = 5 - 2x$$

Square again:

$$4x(x-1) = 25 - 20x + 4x^2$$
$$4x^2 - 4x = 25 - 20x + 4x^2$$
$$16x = 25$$
$$x = \tfrac{25}{16}$$

20. Daisy bought a mixture of two types of her favorite candies costing 40 cents and 50 cents per ounce respectively. The cost of 50 ounces of her mixture was $21.80. How many ounces of the 50-cent candy were in the mixture?

ANSWER: 18

Solution: Let x be the amount of 40-cent candy in the mixture and y the amount of 50-cent.

We have $x + y = 50$

and $40x + 50y = 2180$

$$40x + 40y + 10y = 2180$$
$$(40)(50) + 10y = 2180$$
$$10y = 2180 - 2000 = 180$$
$$y = \underline{18}$$

21. One gear turns $33\frac{1}{3}$ times a minute. Another gear turns 45 times a minute. Initially a mark on each gear is pointing due north. After how many seconds will the two gears next have both marks pointing due north?

ANSWER: 36

Solution: $33\frac{1}{3}$ times in 60 seconds, or

100 times in 180 seconds or

1 time in $\frac{180}{100} = \frac{9}{5}$ seconds.

45 times in 60 seconds or

1 time in $\frac{60}{45} = \frac{4}{3}$ second.

We seek positive integers M and N such that

$$\frac{9}{5}M = \frac{4}{3}N$$
$$27M = 20N$$

Considering prime factorization, the smallest pair (M, N) is $M = 20$ and $N = 27$.

Then $\frac{9}{5}M = \frac{9}{5}20 = 36$ seconds.

$\frac{4}{3}N = \frac{4}{3}27 = \underline{36}$