CD Exam (with solutions)
Texas A&M High School Math Contest
October 20, 2018

1. For any positive integer n let $D(n)$ denote the sum of its digits (in decimal notation). Find all integers n such that $n + 3D(n) = 2018$.

Answer: 1949; 2003.

Since $n \leq 2018$, the number n has at most four digits. Then $D(n) \leq 4 \cdot 9 = 36$. Hence $n = 2018 - 3D(n) \geq 2018 - 3 \cdot 36 = 1910$. It follows that the decimal representation of n is either $20d_1d_0$ or $19d_1d_0$, where d_0 and d_1 are some decimal digits. In the first case, $n = 2000 + 10d_1 + d_0$ and $D(n) = 2 + d_1 + d_0$. Then $n + 3D(n) = 2006 + 13d_1 + 4d_0$. The latter equals 2018 if and only if $13d_1 + 4d_0 = 12$. This is possible only if $d_1 = 0$ and $d_0 = 3$.

In the second case, $n = 1900 + 10d_1 + d_0$ and $D(n) = 10 + d_1 + d_0$. Then the equation $n + 3D(n) = 2018$ is reduced to $13d_1 + 4d_0 = 88$. Note that $52 = 88 - 4 \cdot 9 \leq 13d_1 \leq 88$. We obtain that $4 \leq d_1 \leq 6$. Moreover, $13d_1 = 4(22 - d_0)$, which implies that d_1 is divisible by 4. Hence $d_1 = 4$, then $d_0 = 9$.

2. A 5×5 square drawn on the square grid is then cut into several rectangles of distinct areas (all cuts go along the grid lines). What is the maximal possible number of pieces in such a partition?

Answer: 6.

Let us cut the 5×5 square into one 2×5 rectangle and three 1×5 rectangles. Then we cut one of 1×5 rectangles into 1×1 and 1×4 pieces. Another 1×5 rectangle is cut into 1×2 and 1×3 pieces. Now we have six rectangles of dimensions 1×1, 1×2, 1×3, 1×4, 1×5 and 2×5. Their respective areas are 1, 2, 3, 4, 5 and 10 (where the unit area is that of a 1×1 square).

There are other ways to cut the 5×5 square into six pieces of distinct areas. For example, one can have rectangles of dimensions 1×1, 1×2, 1×3, 1×4, 2×3 and 3×3.

It is not possible to cut the given square into seven or more rectangles of distinct areas. Indeed, the area of each piece is integer. If we had seven or more pieces, their total area would be at least $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. However the area of the 5×5 square is only 25.

3. How many ways there are to change a \$20 bill into smaller bills? (You can use \$1, \$2, \$5 and \$10 bills; no coins are allowed.)

Answer: 40 different ways.

First let us determine which combinations of 5s and 10s can be used, then we add the rest with 1s and 2s. Using only \$5 and \$10 bills, there are three ways to get the full \$20 ($10 + 10$, $10 + 5 + 5$ or $5 + 5 + 5 + 5$), two ways to get \$15 ($10 + 5$ or $5 + 5 + 5$), two ways to get \$10 (10 or $5 + 5$) and one way to get \$5. Also, we can choose to use no 5s or 10s at all. It follows that the desired number is $3 + 2N(5) + 2N(10) + N(15) + N(20)$, where $N(n)$ is the number of ways to split n dollars into 1s and 2s. Note that the number of 2s in such a split can be any whole number between 0 and $n/2$ (inclusive), and once we know it, the number of 1s is determined uniquely. Therefore $N(n) = n/2 + 1$ if n is even and $N(n) = (n + 1)/2$ if n is odd. Hence

$$3 + 2N(5) + 2N(10) + N(15) + N(20) = 3 + 2 \cdot 3 + 2 \cdot 6 + 8 + 11 = 40.$$

4. Given an equilateral triangle ABC with side 1, one chooses a point A_1 on the side BC, a point B_1 on the side AC, and a point C_1 on the side AB. What is the maximal possible perimeter of the triangle $A_1B_1C_1$?

Answer: 3.

By the triangle inequality, $|A_1B_1| \le |A_1C| + |CB_1|$, $|B_1C_1| \le |B_1A| + |AC_1|$ and $|C_1A_1| \le |C_1B| + |BA_1|$. Adding these inequalities, we obtain

$$|A_1B_1|+|B_1C_1|+|C_1A_1| \le |AC_1|+|C_1B|+|BA_1|+|A_1C|+|CB_1|+|B_1A| = |AB|+|BC|+|CA|.$$

Hence the perimeter of the triangle $A_1B_1C_1$ is less than or equal to that of the triangle ABC, which is 3. The equality is attained when the two triangles coincide (for example, when $A_1 = B$, $B_1 = C$ and $C_1 = A$).

5. Solve for x the equation $\big|3 - 2|x|\big| = x + 1$.

Answer: 2/3; 4.

First consider the case $x \ge 0$. In this case the equation is equivalent to $|3-2x| = x+1$. Notice that $3 - 2x = 0$ for $x = 3/2$ so we are going to consider two subcases: $0 \le x < 3/2$ and $x \ge 3/2$. If $0 \le x < 3/2$, then the equation is simplified to $3 - 2x = x + 1$, which has solution $x = 2/3$. If $x \ge 3/2$, then the equation is simplified to $-(3 - 2x) = x + 1$, which has solution $x = 4$.

Now consider the case $x < 0$. In this case the equation is equivalent to $|3 + 2x| = x + 1$. Since $3 + 2x = 0$ for $x = -3/2$, there are two subcases: $-3/2 \le x < 0$ and $x < -3/2$. If $-3/2 \le x < 0$, then the equation is simplified to $3+2x = x+1$, which has solution $x = -2$. However $x = -2$ is not a solution of the original equation since it does not belong to the interval $[-3/2, 0)$. If $x < -3/2$, then the equation is simplified to $-(3 + 2x) = x + 1$, which has solution $x = -4/3$. This one has to be dropped too since it does not belong to the interval $(-\infty, -3/2)$.

6. Find the last digit of the number $2018^{2017^{2016}}$.

Answer: 8.

The last digit equals the remainder under division by 10. The remainder under division of a product by any integer depends only on the remainders of factors. Hence the last digit of 2018^2 coincides with that of $8^2 = 64$. Then the last digit of $2018^3 = 2018^2 \cdot 2018$ coincides with that of $4 \cdot 8 = 32$. Similarly, 2018^4 has the same last digit as $2 \cdot 8 = 16$ while 2018^5 has the same last digit as $6 \cdot 8 = 48$. As we keep multiplying by 2018, the last digits keep repeating: $8, 4, 2, 6, 8, 4, 2, 6, \dots$ This is a periodic sequence with period 4. Therefore the last digit of any power 2018^n depends only on the remainder of n under division by 4.

The number 2017 leaves remainder 1 under division by 4. It follows that any power of 2017 also leaves remainder 1 under division by 4. Thus the last digit of $2018^{2017^{2016}}$ is the same as that of 2018^1.

7. Consider a right triangle ABC with $\angle A = 90^{\circ}$. Let D be the midpoint of the side BC and E be a point on the side AB such that $\angle AEC = \angle BED$. Find the ratio $\dfrac{|AE|}{|EB|}$.

Answer: 1 : 2.

Let F be the image of the vertex C under reflection about the line AB. Since $\angle BAC = 90^o$, the point A is the midpoint of the segment CF. Note that $\angle AEC = \angle AEF$, which implies that $\angle AEF = \angle BED$. Since the segments AE and EB lie on the same line, it follows that the segments FE and ED also lie on the same line. Hence E is the point of intersection of segments AB and DF. Observe that AB and DF are medians of the triangle BCF. It follows that $|BE| : |EA| = 2 : 1$ (as well as $|FE| : |ED| = 2 : 1$).

8. The equation $x^4 + 4x^3 - 3x^2 + 4x + 1 = 0$ has two real solutions. Find their sum.

Answer: -5.

Since $x = 0$ is not a solution, we can divide both sides of the equation by x^2:

$$x^2 + 4x - 3 + \frac{4}{x} + \frac{1}{x^2} = 0.$$

Let $y = x + x^{-1}$. Then $y^2 = x^2 + 2 + x^{-2}$ so that $x^2 + x^{-2} = y^2 - 2$. Hence our equation is equivalent to $y^2 - 2 + 4y - 3 = 0$ or $y^2 + 4y - 5 = 0$. This quadratic equation in y has two roots, 1 and -5. Therefore $x + x^{-1} = 1$ or $x + x^{-1} = -5$. The first of the two equations is equivalent to the quadratic equation $x^2 - x + 1 = 0$, which has no real roots. The second one is equivalent to the equation $x^2 + 5x + 1 = 0$, which has roots $\frac{1}{2}(-5 - \sqrt{21})$ and $\frac{1}{2}(-5 + \sqrt{21})$. Their sum equals -5.

Alternative solution: Since $(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$, it follows that

$$\begin{aligned} x^4 + 4x^3 - 3x^2 + 4x + 1 &= (x + 1)^4 - 9x^2 = \big((x + 1)^2 - 3x\big)\big((x + 1)^2 + 3x\big) \\ &= (x^2 - x + 1)(x^2 + 5x + 1). \end{aligned}$$

The original equation is now reduced to two quadratic equations, $x^2 - x + 1 = 0$ and $x^2 + 5x + 1 = 0$. The first of the two has no real roots. The second one has roots $\frac{1}{2}(-5 - \sqrt{21})$ and $\frac{1}{2}(-5 + \sqrt{21})$. Their sum equals -5.

9. A positive integer m can be represented as $2^4 p_1 p_2 p_3$, where p_1, p_2, p_3 are some odd prime numbers (not necessarily distinct). The integer $m + 100$ can be represented as $5q_1 q_2 q_3$, where q_1, q_2, q_3 are prime numbers different from 5 (not necessarily distinct). The integer $m + 200$ can be represented as $23 r_1 r_2 r_3 r_4$, where r_1, r_2, r_3, r_4 are prime numbers different from 23 (not necessarily distinct). Find m.

Answer: 720.

Since the numbers $m + 100$ and 100 are divisible by 5, so is their sum $m + 200$. Since the number m is divisible by $2^4 = 16$ while 200 is divisible by 8, it follows that the sum $m + 200$ is divisible by $8 = 2^3$. Recall that $m + 200 = 23 r_1 r_2 r_3 r_4$, where r_1, r_2, r_3, r_4 are prime numbers. By the above at least three of the factors r_1, r_2, r_3, r_4 should be equal to 2 and at least one should be equal to 5. This is only possible if the number $m + 200$ is exactly $2^3 \cdot 5 \cdot 23 = 920$. Then $m + 100 = 820 = 2^2 \cdot 5 \cdot 41$ and $m = 720 = 2^4 \cdot 3^2 \cdot 5$.

10. Solve for x the inequality $\sqrt{x - 1} \leq 4 - 3x - x^2$.

Answer: 1.

If $4 - 3x - x^2 < 0$ then x cannot be a solution as $\sqrt{x - 1}$ is nonnegative whenever it is defined. We find that $4 - 3x - x^2 = -(x - 1)(x + 4)$. Hence $4 - 3x - x^2 \geq 0$ if and only if $-4 \leq x \leq 1$.

Since $\sqrt{x-1}$ is well defined only for $x \geq 1$, the only possible solution of the inequality is $x = 1$. One can check that this is indeed a solution.

11. Three points A, B and C divide a circle of radius 1 into three arcs of equal length. We draw three more circles, each intersecting the original circle at two of the three points. All circles intersect at right angles. Find the area of the curvilinear triangle ABC bounded by new circles.

Answer: $3\sqrt{3} - \dfrac{3\pi}{2}$.

Let O be the center of the original circle. The radii OA, OB, and OC divide the curvilinear triangle into three congruent parts. Let S be the area of one of the parts, then the area of the entire triangle is $3S$.

Let O_1 be the center of the circle that intersects the original circle at A and B. It is easy to see that $S = S_1 - S_2$, where S_1 is the area of the quadrilateral OAO_1B and S_2 is the area of the circular sector bounded by segments O_1A, O_1B and the arc of the circle with center O_1 that lies inside the circle with center O. Since the circles intersect at right angles, the angles $\angle A$ and $\angle B$ of the quadrilateral OAO_1B are right. Then the diagonal OO_1 divides the quadrilateral into two right triangles with legs 1 and r, where $r = |O_1A|$. It follows that $S_1 = r$. Besides, $S_2 = (\angle AO_1B/360^{\circ})\pi r^2$.

It remains to find r and $\angle AO_1B$. Clearly, $\angle AOB = \frac{1}{3} \cdot 360^{\circ} = 120^{\circ}$. All angles of the quadrilateral OAO_1B should add up to 360°, therefore $\angle AO_1B = 60^{\circ}$. As a consequence, AO_1B is an equilateral triangle. In particular, $|AB| = r$. The triangle AOB is isosceles. Since $\angle AOB = 120^{\circ}$, the other two angles are equal to 30°. By the law of sines, $|AB|/\sin\angle AOB = |OA|/\sin\angle OBA$ or, equivalently, $r/(\sqrt{3}/2) = 1/(1/2)$. Hence $r = \sqrt{3}$. Now $S_1 = \sqrt{3}$ and $S_2 = \frac{1}{6}\pi r^2 = \frac{\pi}{2}$. Then $S = \sqrt{3} - \frac{\pi}{2}$ and the area of the curvilinear triangle is $3S = 3\sqrt{3} - \frac{3\pi}{2}$.

12. Find all triples (x, y, z) satisfying the system

$$\begin{cases} xyz = 1, \\ x^4yz^2 = 4, \\ x^{10}yz^6 = 16. \end{cases}$$

Answer: $(2, 1, 1/2)$; $(-2, 1, -1/2)$.

From the first equation $y = x^{-1}z^{-1}$. Substituting this into the second equation, we obtain $x^4x^{-1}z^{-1}z^2 = 4$, then $z = 4x^{-3}$. Consequently, $y = x^{-1}z^{-1} = x^{-1}(4x^{-3})^{-1} = x^2/4$. Substituting $y = x^2/4$ and $z = 4x^{-3}$ into the third equation, we get $x^{10}(x^2/4)(4x^{-3})^6 = 16$, which simplifies to $4^5x^{-6} = 16$ and then to $x^6 = 64$. Hence $x = 2$ or $x = -2$. The corresponding values of y and z are found from the relations $y = x^2/4$ and $z = 4x^{-3}$.

Alternative solution: Clearly, x, y and z are all different from zero. Dividing the second equation by the first yields $x^3z = 4$. Dividing the third equation by the second yields $x^6z^4 = 4$. Hence $x^3z = x^6z^4$, which implies that $z = x^{-1}$. Substituting this into the first equation yields $y = 1$. Substituting $y = 1$ and $z = x^{-1}$ into the second equation yields $x^2 = 4$. Thus $x = 2$ or $x = -2$, then $z = 1/2$ or $z = -1/2$, respectively.

13. Find all pairs of integers (x, y) satisfying the equation $2^x - 3^y = 1$.

Answer: $(1, 0)$; $(2, 1)$.

If $x \le 0$ then $2^x - 3^y \le 1 - 3^y < 1$. Therefore x must be positive. Since the equation is equivalent to $2^x - 1 = 3^y$, for any $x > 0$ there is a unique y that, together with x, makes a solution. However the solution need not be integer. If $x = 1$ then $y = 0$. If $x = 2$ then $y = 1$.

Let us show that there are no more integer solutions. Assume the contrary: $2^x - 3^y = 1$ for some integers x and y, where $x \ge 3$. Then $3^y = 2^x - 1 \ge 2^3 - 1 = 7$, which implies that $y \ge 2$. As a consequence, the number 2^x leaves remainder 1 under division by $3^2 = 9$. Consecutive powers of 2 leave the following remainders under division by 9: $2^1 \equiv 2 \bmod 9$, $2^2 \equiv 4 \bmod 9$, $2^3 \equiv 8 \bmod 9$, $2^4 \equiv 7 \bmod 9$, $2^5 \equiv 5 \bmod 9$, $2^6 \equiv 1 \bmod 9$, then they start repeating: $2^7 \equiv 2 \bmod 9$ and so on. Hence the sequence of remainders is periodic with period 6, and $2^x \equiv 1 \bmod 9$ only if x is divisible by 6. Therefore $x = 6k$ for some positive integer k. Then

$$2^x - 1 = 2^{6k} - 1 = (2^6)^k - 1 = 64^k - 1 = (64 - 1)(1 + 64 + 64^2 + \cdots + 64^{k-1}).$$

In particular, the number $2^x - 1$ is divisible by $64 - 1 = 63$. Since $63 = 7 \cdot 9$, it is divisible by 7 as well. But then $2^x - 1$ cannot be equal to 3^y, a contradiction.

14. Solve for x the equation $\sqrt{6 + \sqrt{6 + \sqrt{6 + x}}} = x$.

Answer: 3.

The equation can be rewritten as $f(f(f(x))) = x$, where $f(x) = \sqrt{6 + x}$. The function f, which is defined for $x \ge -6$, is strictly increasing on its entire domain. Suppose $f(f(f(x)))$ is defined for some x. If $f(x) > x$, then $f(f(x)) > f(x)$ and $f(f(f(x))) > f(f(x))$ so that $f(f(f(x))) > x$. If $f(x) < x$, then $f(f(x)) < f(x)$ and $f(f(f(x))) < f(f(x))$ so that $f(f(f(x))) < x$. On the other hand, if $f(x) = x$, then $f(f(f(x))) = x$ as well. It follows that the equation $f(f(f(x))) = x$ is equivalent to the equation $f(x) = x$. Squaring both sides of the equation $\sqrt{6 + x} = x$, we obtain a quadratic equation $6 + x = x^2$, which has roots -2 and 3. Only $x = 3$ is a solution of the original equation.

15. Let ABC be an isosceles triangle with the base AB. Let AD be the median and AE be the angle bisector of this triangle. Find the length of the leg AC if $|AB| = 5$ and $|DE| = 6$.

Answer: 20.

The angle bisector AE divides the side BC of the triangle ABC into two segments whose lengths are proportional to lengths of the other two sides:

$$\frac{|EB|}{|AB|} = \frac{|EC|}{|AC|}.$$

Note that the segment DE, which is part of the leg BC, is longer than the base AB. It follows that $|AC| > |AB|$, then $|EC| > |EB|$. As a consequence, the point E lies between the points D and B. Let $|AC| = x$. Then $|DB| = |DC| = x/2$, $|EB| = |DB| - |DE| = x/2 - 6$, and $|EC| = |DC| + |DE| = x/2 + 6$. We obtain that

$$\frac{x/2 - 6}{5} = \frac{x/2 + 6}{x}.$$

Multiplying both sides of this equation by $10x$, we get $x(x-12) = 5(x+12)$, which is simplified to $x^2 - 17x - 60 = 0$. The quadratic equation has roots 20 and -3. Only the positive root makes sense here.

16. Solve for x the equation $\log_2 x \cdot \log_2(x-2) + 1 = \log_2(x^2 - 2x)$.

Answer: 4.

Let $y = \log_2 x$ and $z = \log_2(x-2)$. Then $\log_2(x^2-2x) = \log_2 x + \log_2(x-2) = y + z$ and the equation becomes $yz+1 = y+z$. This is equivalent to $yz - y - z + 1 = 0$ or $(y-1)(z-1) = 0$. Hence $y = 1$ or $z = 1$. The equation $\log_2 x = 1$ has solution $x = 2$. The equation $\log_2(x-2) = 1$ has solution $x = 4$. Only $x = 4$ is a solution of the original equation since $\log_2(x-2)$ and $\log_2(x^2-2x)$ are not defined for $x = 2$.

17. Find all polynomials $P(x)$ such that $(P(x))^2 = P(x-1)P(x+1)$ for all x.

Answer: all constant polynomials.

Clearly, every constant polynomial satisfies the functional equation. Now assume $P(x)$ is a nonconstant polynomial. Then the equation $P(x) = 0$ has only finitely many roots. If we allow complex roots, then at least one root does exist. Let z_0 be the root with the maximal possible real part. Then $z_1 = z_0 + 1$ is not a root. In particular, $(P(z_1))^2 \neq 0$. However $P(z_1 - 1)P(z_1 + 1) = P(z_0)P(z_1 + 1) = 0$. Thus the polynomials $(P(x))^2$ and $P(x-1)P(x+1)$ cannot be the same.

18. In an acute triangle ABC with the median AD and the altitude AE, the point F at which the inscribed circle touches the side BC is the midpoint of the segment DE. Find the perimeter of the triangle ABC if $|BC| = 1$.

Answer: 3.

The sides AB and AC have different lengths as otherwise the points D and E would be the same. Without loss of generality, assume that $|AB| > |AC|$. Then the segment DE is part of the segment DC.

The altitude AE cuts the triangle ABC into two right triangles. By the Pythagorean theorem, $|AB|^2 = |AE|^2 + |BE|^2$ and $|AC|^2 = |AE|^2 + |CE|^2$. It follows that $|AB|^2 - |AC|^2 = |BE|^2 - |CE|^2$. Note that $|BE| + |CE| = |BC| = 1$ and $|BE| - |CE| = 2|DE|$. Hence $|AB|^2 - |AC|^2 = 2|DE|$.

Let G and H be points where the inscribed circle touches sides AB and AC, respectively. Then $|AG| = |AH|$, $|BG| = |BF|$, and $|CH| = |CF|$. Therefore

$$|BF| - |CF| = |BG| - |CH| = |BG| + |AG| - |AH| - |CH| = |AB| - |AC|.$$

On the other hand, $|BF| - |CF| = 2|DF|$. Hence $|AB| - |AC| = 2|DF|$. Recall that F is the midpoint of the segment DE. Then $2|DF| = |DE|$. Finally,

$$2|DE| = |AB|^2 - |AC|^2 = (|AB| - |AC|)(|AB| + |AC|) = |DE|\,(|AB| + |AC|),$$

which implies that $|AB| + |AC| = 2$. Thus the perimeter of the triangle ABC is $|AB| + |AC| + |BC| = 2 + 1 = 3$.

19. Evaluate the sum

$$\frac{1}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \frac{3}{4 \cdot 5 \cdot 6} + \cdots + \frac{2015}{2016 \cdot 2017 \cdot 2018}.$$

Answer: $\dfrac{1}{4}+\dfrac{1}{2\cdot 2017}-\dfrac{3}{2\cdot 2018}=\dfrac{507780}{2035153}.$

Each term in the sum is of the form $\frac{n}{(n+1)(n+2)(n+3)}$, where n runs from 1 to 2015. We transform it as follows:

$$\frac{n}{(n+1)(n+2)(n+3)}=\frac{n}{n+1}\cdot\frac{1}{(n+2)(n+3)}=\left(1-\frac{1}{n+1}\right)\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$$
$$=\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+3)}$$
$$=\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+3}\right).$$

Now the sum decomposes into three telescopic sums:

$$\sum_{n=1}^{2015}\frac{n}{(n+1)(n+2)(n+3)}=\sum_{n=1}^{2015}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\sum_{n=1}^{2015}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$
$$+\frac{1}{2}\sum_{n=1}^{2015}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)=\left(\frac{1}{3}-\frac{1}{2018}\right)-\left(\frac{1}{2}-\frac{1}{2017}\right)+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{2017}-\frac{1}{2018}\right)$$
$$=\frac{1}{4}+\frac{1}{2\cdot 2017}-\frac{3}{2\cdot 2018}=\frac{2017\cdot 1009+2018-3\cdot 2017}{2\cdot 2017\cdot 2018}=\frac{2031120}{4\cdot 2017\cdot 1009}=\frac{507780}{2035153}.$$

20. Solve for x the equation $\sqrt[3]{2x-1}-\sqrt[3]{x-1}=1$.

Answer: 1; $28+6\sqrt{21}$; $28-6\sqrt{21}$.

Let $y=\sqrt[3]{2x-1}$ and $z=\sqrt[3]{x-1}$. Then $y^3=2x-1$, $z^3=x-1$, and the given equation is equivalent to $y-z=1$. Hence every solution of the equation is part of a solution of the system

$$\begin{cases} y^3=2x-1\\ z^3=x-1\\ y-z=1\end{cases}$$

and vice versa. Let us solve the second equation of the system for x, solve the third equation for y, and then substitute them into the first equation. This yields

$$\begin{cases} y^3=2x-1\\ x=z^3+1\\ y=z+1\end{cases}\iff\begin{cases} (z+1)^3=2(z^3+1)-1\\ x=z^3+1\\ y=z+1\end{cases}\iff\begin{cases} z^3-3z^2-3z=0\\ x=z^3+1\\ y=z+1\end{cases}$$

The roots of the cubic polynomial $z^3-3z^2-3z=z(z^2-3z-3)$ are 0 and $\frac{1}{2}(3\pm\sqrt{21})$. The corresponding solutions of the original equation are found from the relation $x=z^3+1$. Thus $x=1$ or

$$x=\frac{(3\pm\sqrt{21})^3}{8}+1=\frac{27\pm 27\sqrt{21}+9\cdot 21\pm 21\sqrt{21}}{8}+1=\frac{216\pm 48\sqrt{21}}{8}+1=28\pm 6\sqrt{21}.$$

21. Let $a_1, a_2, \ldots, a_{2018}$ be the numbers $1, 2, \ldots, 2018$ written in some order. What is the maximal possible value of $a_1a_2 + a_3a_4 + \cdots + a_{2017}a_{2018}$? The following equalities may (or may not) help to calculate the value:

$$1 + 2 + 3 + 4 + \cdots + 2017 + 2018 = 2037171,$$
$$-1^2 + 2^2 - 3^2 + 4^2 - \cdots - 2017^2 + 2018^2 = 2037171,$$
$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + 2017^2 + 2018^2 = 2741353109.$$

Answer: $(1^2 + 2^2 + 3^2 + \cdots + 2018^2 - 1009)/2 = 1370676050.$

Let $X = a_1a_2 + a_3a_4 + \cdots + a_{2017}a_{2018}$ and $Y = (a_1 - a_2)^2 + (a_3 - a_4)^2 + \cdots + (a_{2017} - a_{2018})^2$. We have $(a_n - a_{n+1})^2 = a_n^2 + a_{n+1}^2 - 2a_na_{n+1}$ for all n. Summing this up over all odd integers n from 1 to 2017, we obtain that $Y = A - 2X$, where $A = a_1^2 + a_2^2 + \cdots + a_{2018}^2$. It is easy to see that $A = 1^2+2^2+\cdots+2018^2$, in particular, this is always the same number. Therefore the sum X attains its maximal value exactly when Y attains its minimal value. Each term in the sum Y is the square of a nonzero integer. It follows that the minimal possible value of Y is $2018/2 = 1009$, attained when each term equals 1. This happens, for example, when the numbers are not rearranged at all: $a_n = n$ for all n. Using the formula

$$1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6},$$

we find that $A = 2018 \cdot 2019 \cdot 4037/6 = 1009 \cdot 673 \cdot 4037 = 2741353109$. Then the maximal value of $X = (A - Y)/2$ is $(2741353109 - 1009)/2 = 1370676050$.

22. A sequence of real numbers $a_1, a_2, a_3, \ldots$ is built recursively using the rule $a_{n+2} = a_na_{n+1} + 1$ for $n = 1, 2, \ldots$ The sequence happens to be periodic, that is, $a_{n+k} = a_n$ for some $k \geq 1$ and all n. Let s_n be the sign of the number a_n (s_n is $+$ if $a_n > 0$ and $-$ if $a_n < 0$; for $a_n = 0$ it can be either $+$ or $-$). Assuming s_1 is $+$, find all possible sequences $s_1, s_2, s_3, \ldots$ In each case, describe the sequence of signs completely or list at least the first 10 terms. (Hint: since the sequence $a_1, a_2, a_3, \ldots$ is periodic, some simple combinations of signs cannot occur in $s_1, s_2, s_3, \ldots$)

Answer: only one possibility: $+ - - + - - + - - + - - \ldots$
(An example of the periodic recursive sequence is $2, -1, -1, 2, -1, -1, \ldots$)

First we shall show that the combination $++$ cannot occur in the sequence of signs. Assume the contrary, that is, $a_n \geq 0$ and $a_{n+1} \geq 0$ for some n. Then $a_{n+2} = a_na_{n+1} + 1 \geq 1$ and $a_{n+3} = a_{n+1}a_{n+2} + 1 \geq 1$. Next $a_{n+4} = a_{n+2}a_{n+3} + 1 \geq a_{n+3} + 1 > a_{n+3} \geq 1$. Now it follows by induction that the sequence $a_{n+3}, a_{n+4}, a_{n+5}, \ldots$ is strictly increasing. However this contradicts with the fact that the sequence $a_1, a_2, a_3, \ldots$ is periodic.

Next observation is that there are no zeros in the sequence $a_1, a_2, \ldots$ Indeed, if $a_n = 0$ for some $n \geq 2$, then $a_{n+1} = a_{n-1}a_n + 1 = 1$ and $a_{n+2} = a_na_{n+1} + 1 = 1$, which is not possible as shown above. Besides, a_1 cannot be the lone zero since the sequence is periodic.

Now let us show that the combination $+-+$ cannot occur in the sequence of signs. Assume the contrary, that is, $a_n > 0$, $a_{n+1} < 0$ and $a_{n+2} > 0$ for some n. Since the sequence $a_1, a_2, \ldots$ is periodic, the number n can be chosen arbitrarily large. Notice that $a_{n+2} = a_na_{n+1} + 1 < 1$. By the above every positive number must be followed by a negative one, hence $a_{n+3} < 0$. In particular,

$a_{n+3} < a_{n+2}$, which implies that $a_{n+1}a_{n+2} < a_n a_{n+1}$. Since $a_{n+1} < 0$, it follows that $a_n < a_{n+2}$. Now additionally assume that $n \geq 3$. Then $a_{n-1} < 0$ (since $a_n > 0$) and $a_{n-2} > 0$ (as otherwise $a_n = a_{n-2}a_{n-1} + 1 > 1 > a_{n+2}$). That is, the combination of signs $+-+$ is preceded by $+-$. It follows by induction that $a_{n+2}, a_n, a_{n-2}, a_{n-4}, \ldots$ is a strictly decreasing sequence of positive numbers. This sequence is finite (it ends with a_1 or a_2) but it can be arbitrarily long (depending on the choice of n) so this does contradict with the periodicity of the sequence $a_1, a_2, a_3, \ldots$

We have shown that the sequence $a_1, a_2, \ldots$ contains no zeros and that the sequence of signs does not contain combinations $++$ and $+-+$. Hence every positive number must be followed by two negative numbers. On the other hand, if a_n and a_{n+1} are both negative for some n, then $a_{n+2} = a_n a_{n+1} + 1 > 1$ is positive. This leaves us the only possibility for the sequence of signs: $+--+--+--\ldots$