DE Exam Solutions
Texas A&M High School Math Contest
October 20, 2018

All answers must be simplified, and if units are involved, be sure to include them.

1. Solve the equation $4^{x-3} - 8^{x+5} = 0$.

   Solution: The equation is equivalent to

$$2^{2(x-3)} = 2^{3(x+5)} \Leftrightarrow 2x - 6 = 3x + 15 \Leftrightarrow x = -21.$$

   Answer: -21

2. Find the value of $\frac{y}{z}$ if $3wz + 4xy - 2wy - 6xz = 0$, $w \neq 2x$ and $z \neq 0$.

   Solution: We have that

$$3wz + 4xy - 2wy - 6xz = 0 \Leftrightarrow 3z(w - 2x) - 2y(w - 2x) = 0 \Leftrightarrow (w - 2x)(3z - 2y) = 0.$$

   Since $w \neq 2x$ and $z \neq 0$ it implies that

$$3z - 2y = 0 \Leftrightarrow 3z = 2y \Leftrightarrow \frac{y}{z} = \frac{3}{2}.$$

   Answer: $\frac{3}{2}$

3. If $\log x + \log y = \frac{29}{10}$ and $\log x \log y = 1$ find the value of

$$\log_x y + \log_y x.$$

   Solution: We can write our expression as

$$\log_x y + \log_y x = \frac{\log y}{\log x} + \frac{\log x}{\log y} = \frac{\log^2 y + \log^2 x}{\log x \log y}.$$

   Using this and the hypotheses we obtain that

$$\log_x y + \log_y x = \log^2 y + \log^2 x + 2\log x \log y - 2 = (\log x + \log y)^2 - 2 = \left(\frac{29}{10}\right)^2 - 2 = \frac{641}{100}.$$

   Answer: $\frac{641}{100}$ or 6.41

4. Let x be a real number and y be a positive integer such that $x > 1$ and $\frac{x}{3} = \frac{5x+1}{3y+2}$. Find y.

   Solution: Since y is a positive integer, the equation is equivalent to

$$3xy + 2x = 15x + 3 \Leftrightarrow x(3y - 13) = 3 \Leftrightarrow x = \frac{3}{3y - 13}.$$

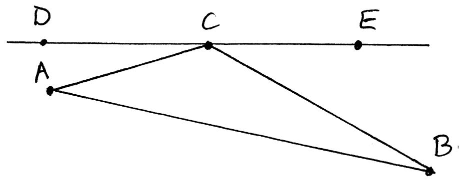
   The condition $x > 1$ implies

$$\frac{3}{3y - 13} > 1 \Leftrightarrow \frac{16 - 3y}{3y - 13} > 0 \Leftrightarrow y \in \left(\frac{13}{3}, \frac{16}{3}\right).$$

   Since y is a positive integer we get that $y = 5$.

   Answer: 5

5. In the figure below we have $AC = 2$, $BC = 3$, $\angle DCA = 15°$, and $\angle ECB = 30°$. Find AB.



Solution: Since the points D, C, and E are collinear, it implies that

$$\angle DCA + \angle ACB + \angle ECB = 180° \Leftrightarrow \angle ACB = 180° - 15° - 30° = 135°.$$

From the law of cosines we have that

$$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cos(\angle ACB) = 13 - 12\cos 135° = 13 + 6\sqrt{2}.$$

Answer: $\sqrt{13 + 6\sqrt{2}}$

6. The probability that a worker with occupational exposure to dust contracts a lung disease is $\frac{1}{6}$. Three such workers are checked at random. Find the probability that at least one of them contracted a lung disease.

Solution: We use that

$$P(\text{at least one of them contracted a lung disease}) = 1 - P(\text{none of them contracted a lung disease})$$

and that

$$P(\text{a worker does not contract a lung disease}) = 1 - P(\text{a worker contracts a lung disease}) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Since we are dealing with independent events here, we get that

$$P(\text{none of them contracted a lung disease}) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216},$$

and therefore,

$$P(\text{at least one of them contracted a lung disease}) = 1 - \frac{125}{216} = \frac{91}{216}.$$

Answer: $\frac{91}{216}$

7. Find the value of $\tan 1° \tan 2° \tan 3° \cdots \tan 88° \tan 89°$.

Solution: Using the cofunction identity $\tan\theta = \cot(90° - \theta)$ and the inverse identity $\cot\theta = \frac{1}{\tan\theta}$ we can write

$$\tan 1° \tan 2° \tan 3° \cdots \tan 88° \tan 89° = \tan 1° \tan 2° \cdots \tan 44° \tan 45° \tan 46° \cdots \tan 88° \tan 89° =$$
$$\tan 1° \tan 2° \cdots \tan 44° \cdot 1 \cdot \cot 44° \cdots \cot 2° \cot 1° = (\tan 1° \cot 1°)(\tan 2° \cot 2°) \cdots (\tan 44° \cot 44°) = 1.$$

Answer: 1

8. Find xy, where x and y satisfy the system of equations

$$\begin{cases} \dfrac{1}{x-2}+\dfrac{1}{y} &= 6 \\ 4x+47y-22xy &= 8. \end{cases}$$

Solution: x and y must satisfy the conditions $x \neq 2$ and $y \neq 0$. The first equation can be written as

$$y+x-2 = 6y(x-2) \Leftrightarrow x+13y-6xy = 2 \Leftrightarrow 4x+52y-24xy = 8.$$

By subtracting the second equation from the one above we get $5y-2xy = 0 \Leftrightarrow (5-2x)y = 0$ which implies, since $y \neq 0$, that $x = \dfrac{5}{2}$. From the first equation we can find that $y = \dfrac{1}{4}$. Therefore, $xy = \dfrac{5}{8}$.

Answer: $\dfrac{5}{8}$

9. Find the real number k such that the equation $|x^2-2x-8| = k$ has exactly three real distinct solutions.

Solution: The equation is equivalent to

$$x^2-2x-8 = k \text{ or } x^2-2x-8 = -k \Leftrightarrow x^2-2x-(8+k) = 0 \text{ or } x^2-2x-(8-k) = 0.$$

The discriminants for the two quadratic equations are $\Delta_1 = 36+4k$ and $\Delta_2 = 36-4k$. To have exactly three real distinct roots we need one equation to have two real distinct roots and the other one to have repeated roots. So one discriminant is positive and the other one is 0. Since $k \geq 0$, it implies that $36-4k = 0 \Leftrightarrow k = 9$.

Answer: 9

10. Find the coefficient of x^2 in the expansion of $(2-x)^6(1+3x)^7$.

Solution: One can use Pascal's triangle for the binomial expansion of $(a+b)^n$ to write

$$(2-x)^6 = 2^6 + 6 \cdot 2^5(-x) + 15 \cdot 2^4(-x)^2 + \cdots = 64 - 192x + 240x^2 + \cdots$$
$$(1+3x)^7 = 1^7 + 7 \cdot 1^6(3x) + 21 \cdot 1^5(3x)^2 + \cdots = 1 + 21x + 189x^2 + \cdots.$$

Therefore, the coefficient of x^2 in the given expansion is $64 \cdot 189 - 21 \cdot 192 + 240 = 8304$.

Answer: 8304

11. Determine the sum of all integers n such that the number $n^2+9n+14$ is the square of another integer.

Solution: Let k be a nonnegative integer such that $n^2+9n+14 = k^2$. This is equivalent to

$$n^2+9n+\left(\frac{9}{2}\right)^2-\left(\frac{9}{2}\right)^2+14 = k^2 \Leftrightarrow \left(n+\frac{9}{2}\right)^2-k^2 = \frac{25}{4} \Leftrightarrow (2n+9-2k)(2n+9+2k) = 25.$$

The integer divisors of 25 are -1, -5, -25, 1, 5, and 25. Therefore, we have a few cases. The case

$$\begin{cases} 2n+9-2k &= 1 \\ 2n+9+2k &= 25 \end{cases} \Rightarrow 4n+18 = 26 \Leftrightarrow n = 2 \text{ and } n^2+9n+14 = 36.$$

Similarly, the cases

$$\begin{cases} 2n+9-2k &= -25 \\ 2n+9+2k &= -1, \end{cases} \quad \begin{cases} 2n+9-2k &= 5 \\ 2n+9+2k &= 5 \end{cases} \text{ and } \begin{cases} 2n+9-2k &= -5 \\ 2n+9+2k &= -5 \end{cases}$$

give us $n = -11$, $n = -2$, and $n = -7$, respectively. Then the sum of all integers that satisfy our problem is $2+(-11)+(-2)+(-7) = -18$.

Answer: -18

12. Find the maximum value of the expression $(2n^2+3n)\sqrt{3}-(3n^2+2n)\sqrt{2}$, where n is an integer.

Solution. Consider the quadratic function

$$f(x)=(2x^2+3x)\sqrt{3}-(3x^2+2x)\sqrt{2}=(2\sqrt{3}-3\sqrt{2})x^2+(3\sqrt{3}-2\sqrt{2})x.$$

Since $2\sqrt{3}-3\sqrt{2}<0$, it implies that $f(x)$ has a maximum value (x being a real number) at $x=\dfrac{3\sqrt{3}-2\sqrt{2}}{2(3\sqrt{2}-2\sqrt{3})}=\dfrac{6+5\sqrt{6}}{12}\in(1,2)$. Therefore, the maximum value for our expression is $f(1)=5(\sqrt{3}-\sqrt{2})$ or $f(2)=14\sqrt{3}-16\sqrt{2}$. We can prove that $f(2)>f(1)$ since

$$14\sqrt{3}-16\sqrt{2}>5\sqrt{3}-5\sqrt{2}\Leftrightarrow 9\sqrt{3}>11\sqrt{2}\Leftrightarrow 243>242.$$

Answer: $14\sqrt{3}-16\sqrt{2}$

13. Let $P(x)$ be a polynomial of degree at least two such that the remainders for the division of $P(x)$ by $x-3$ and $x+5$ are 5 and -11, respectively. Find the remainder of the division of $P(x)$ by $x^2+2x-15$.

Solution: We have that $P(x)=(x^2+2x-15)Q(x)+R(x)$, where $R(x)=ax+b$ and $x^2+2x-15=(x+5)(x-3)$. So $P(x)=(x+5)(x-3)Q(x)+ax+b$. We know that $P(-5)=-11$ and $P(3)=5$. We obtain the system of equations

$$\begin{cases}-5a+b&=-11\\3a+b&=5,\end{cases}\Leftrightarrow\begin{cases}8a&=16\\3a+b&=5,\end{cases}\Leftrightarrow\begin{cases}a&=2\\b&=-1.\end{cases}$$

Therefore, $R(x)=2x-1$.

Answer: $2x-1$

14. Simplify the fraction

$$\frac{27n^3+6n^2-37n+4}{27n^3-21n^2-70n+8}$$

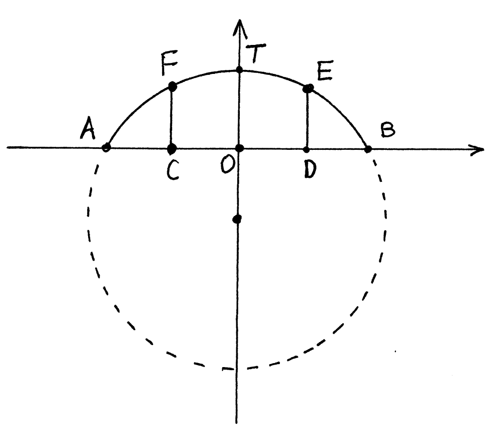
and then find its value for $n=56789$.

Solution: We notice that

$$\frac{27n^3+6n^2-37n+4}{27n^3-21n^2-70n+8}=\frac{(n-1)(27n^2+33n-4)}{(n-2)(27n^2+33n-4)}=\frac{n-1}{n-2}.$$

Answer: $\dfrac{56788}{56787}$

15. Consider cartesian coordinates with the origin at the point O and axes OB and OT. The diagram below shows the arch $AFTEB$ of a stone bridge. The bridge forms an arc of a circle and length AB forms a chord of the circle. AB is 24 feet and the top of the bridge T is 3 feet vertically above AB. C and D are midpoints of OA and OB. CF and DE are two vertical pillars supporting the arch. Find the height of the pillar DE.



Solution: Let (h, k) be the center of the circle and r be the radius of the circle. We see that $h = 0$. So the equation of the circle is $x^2 + (y-k)^2 = r^2$. From the fact that B and T belong to the circle we get the system of equations

$$\begin{cases} 144 + k^2 & = r^2 \\ (3-k)^2 & = r^2, \end{cases} \Leftrightarrow \begin{cases} 144 + k^2 & = (3-k)^2 \\ (3-k)^2 & = r^2, \end{cases} \Leftrightarrow \begin{cases} -6k & = 135 \\ (3-k)^2 & = r^2, \end{cases} \Leftrightarrow \begin{cases} k & = -\frac{45}{2} \\ r & = \frac{51}{2}. \end{cases}$$

The equation of the circle becomes $x^2 + (y + \frac{45}{2})^2 = \frac{2601}{4}$. Since $x_E = x_D = 6$ and E is on the circle we obtain that

$$36 + (y_E + \frac{45}{2})^2 = \frac{2601}{4} \Leftrightarrow y_E + \frac{45}{2} = \frac{\sqrt{2457}}{2} \Leftrightarrow y_E = \frac{\sqrt{2457} - 45}{2}.$$

Answer: $\dfrac{\sqrt{2457} - 45}{2}\ ft$ or $\dfrac{3\sqrt{273} - 45}{2}\ ft$

16. Find the value of $\log_2(x_1x_2)$, where x_1 and x_2 are the solutions of the equation

$$\log_2 x^{\sqrt{5}+1} + \log_x 4^{\sqrt{5}+1} = \log_2(16x^3) - \log_x 16.$$

Solution: For the logarithms to make sense we need $x > 0$, $x \neq 1$. The equation is equivalent to

$$(\sqrt{5}+1)\log_2 x + (\sqrt{5}+1)\log_x 4 = \log_2 16 + \log_2 x^3 - 4\log_x 2 \Leftrightarrow$$
$$(\sqrt{5}+1)\log_2 x + 2(\sqrt{5}+1)\log_x 2 = 4 + 3\log_2 x - 4\log_x 2.$$

If we denote $\log_2 x = y$ then our equation becomes

$$(\sqrt{5}+1)y + 2(\sqrt{5}+1)\frac{1}{y} = 4 + 3y - \frac{4}{y} \Leftrightarrow$$
$$(\sqrt{5}-2)y - 4 + (2\sqrt{5}+6)\frac{1}{y} = 0 \Leftrightarrow (\sqrt{5}-2)y^2 - 4y + (2\sqrt{5}+6) = 0.$$

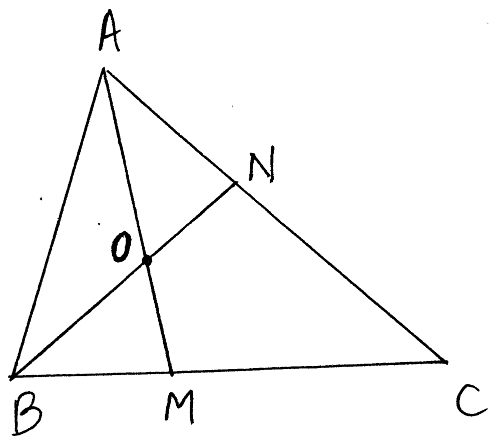
The above quadratic equation in y has two real solutions y_1 and y_2 such that $y_1 + y_2 = \dfrac{4}{\sqrt{5}-2}$. Then if x_1 and x_2 are the solutions of $\log_2 x = y_1$ and $\log_2 x = y_2$, respectively, we have

$$\log_2(x_1x_2) = \log_2 x_1 + \log_2 x_2 = y_1 + y_2 = \frac{4}{\sqrt{5}-2}.$$

Answer: $\dfrac{4}{\sqrt{5}-2}$ or $4(\sqrt{5}+2)$

17. Consider the triangle ABC in which the angle bisector of $\angle A$ intersects side BC at a point M and the angle bisector of $\angle B$ intersects side AC at a point N. Let O be the intersection point between AM and BN. We know that $\dfrac{AO}{OM} = \sqrt{3}$ and $\dfrac{ON}{BO} = \sqrt{3} - 1$. Find $\angle C$.

Solution: Let $AB = c$, $BC = a$, and $AC = b$.



On one hand the ratio between the areas of triangle ABO and triangle MBO is equal to $\frac{OA}{OM}$. On the other hand the same ratio is equal to $\frac{AB \cdot BO \cdot \sin(\angle ABO)}{BM \cdot BO \cdot \sin(\angle MBO)}$. Since BO is the angle bisector of $\angle ABM$, we get that $\frac{OA}{OM} = \frac{c}{BM}$. A similar argument in triangle ABC gives us

$$\frac{BM}{MC} = \frac{c}{b} \Leftrightarrow \frac{BM}{BM + MC} = \frac{c}{b+c} \Leftrightarrow BM = \frac{ac}{b+c}.$$

Therefore,

$$\sqrt{3} = \frac{OA}{OM} = \frac{c}{BM} = \frac{b+c}{a} \Leftrightarrow b + c = \sqrt{3}a.$$

Similarly, $\frac{ON}{OB} = \frac{CN}{a}$ and $CN = \frac{ab}{a+c}$. So, $b = (\sqrt{3} - 1)(a + c)$. From the two relations between a, b, and c we get that $2a = \sqrt{3}b$ and $a = \sqrt{3}c$. From the law of sines we have that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Leftrightarrow \frac{\frac{\sqrt{3}}{2}}{\sin A} = \frac{1}{\sin B} = \frac{\frac{1}{2}}{\sin C} \Leftrightarrow \frac{\sin\frac{\pi}{3}}{\sin A} = \frac{\sin\frac{\pi}{2}}{\sin B} = \frac{\sin\frac{\pi}{6}}{\sin C}.$$

It can be shown that two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar if and only if

$$\frac{\sin A_1}{\sin A_2} = \frac{\sin B_1}{\sin B_2} = \frac{\sin C_1}{\sin C_2}.$$

It implies that $\angle C = \frac{\pi}{6}$.

Answer: $\frac{\pi}{6}$

18. Find the distance from the center to the foci of the hyperbola with vertices $(5, -6)$ and $(5, 6)$, passing through the point $(0, 9)$.

Solution: The center of the hyperbola, (h, k), is the midpoint between the vertices. So $(h, k) = (5, 0)$. Since the transverse axis is vertical, the standard form of the equation of our hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \Leftrightarrow \frac{y^2}{a^2} - \frac{(x-5)^2}{b^2} = 1.$$

a is the distance between the center and the vertices. Therefore, $a = 6$. The equation becomes $\frac{y^2}{36} - \frac{(x-5)^2}{b^2} = 1$. From the fact that the point $(0, 9)$ belongs to the hyperbola we get that

$$\frac{81}{36} - \frac{25}{b^2} = 1 \Leftrightarrow \frac{25}{b^2} = \frac{5}{4} \Leftrightarrow b^2 = 20.$$

The distance from the center to the foci is $c = \sqrt{a^2 + b^2} = \sqrt{56} = 2\sqrt{14}$.

Answer: $\sqrt{56}$ or $2\sqrt{14}$

19. Find $\cot^2 36° \cot^2 72°$.

Solution: We know that $36° = \frac{\pi}{5}$. If we denote $a = \frac{\pi}{5}$ then $\sin 2a = \sin 3a \Leftrightarrow 2\sin a \cos a = 3\sin a - 4\sin^3 a$. Hence $2\cos a = 3 - 4\sin^2 a = 4\cos^2 a - 1$ which implies that $x = \cos a$ satisfies the quadratic equation $4x^2 - 2x - 1 = 0$. Since $\cos a > 0$ we get that $\cos a = \dfrac{1+\sqrt{5}}{4}$. From here we obtain that $\cos^2 a = \dfrac{3+\sqrt{5}}{8}$ and $\sin^2 a = \dfrac{5-\sqrt{5}}{8}$. Next we have that $\cot^2 a = \dfrac{3+\sqrt{5}}{5-\sqrt{5}}$ and $\cot^2 a - 1 = \dfrac{2}{\sqrt{5}}$. Finally,

$$\cot^2 a \cot^2 2a = \cot^2 a \frac{(\cot^2 a - 1)^2}{4\cot^2 a} = \frac{(\cot^2 a - 1)^2}{4} = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5}.$$

Answer: $\dfrac{1}{5}$

20. Find the minimum value of the function

$$f(x) = 1 \cdot |x-1| + 2 \cdot |x-2| + 3 \cdot |x-3| + \cdots + 20 \cdot |x-20|.$$

Solution: If $x \leq 1$ then

$$\begin{aligned} f(x) &= -1(x-1) - 2(x-2) - \cdots - 20(x-20) = -(1+2+\cdots+20)x + 1^2 + 2^2 + \cdots + 20^2 \\ &= -\frac{20 \cdot 21}{2}x + \frac{20(20+1)(2\cdot 20+1)}{6} = -210x + 2870 \geq -210 + 2870 = 2660. \end{aligned}$$

Similarly, if $x \geq 20$ then $f(x) = 210x - 2870 \geq 4200 - 2870 = 1330$. If $x \in [k, k+1]$, $k \in \{1, 2, \cdots, 19\}$ then

$$\begin{aligned} f(x) &= 1\cdot(x-1) + \cdots + k(x-k) - (k+1)(x-k-1) - \cdots - 20(x-20) \\ &= \{1 + 2 + \cdots + k - [(k+1) + \cdots + 20]\}x - 1^2 - 2^2 - \cdots - k^2 + (k+1)^2 + \cdots + 20^2 \\ &= [2(1+2+\cdots+k) - (1+2+\cdots+20)]x + 1^2 + 2^2 + \cdots + 20^2 - 2(1^2 + 2^2 + \cdots + k^2) \\ &= (k^2 + k - 210)x + 2870 - \frac{k(k+1)(2k+1)}{3} = (k+15)(k-14)x + 2870 - \frac{k(k+1)(2k+1)}{3}. \end{aligned}$$

If $x \in [1, 14]$, then there exists $k \in \{1, 2, \cdots, 13\}$ such that $x \in [k, k+1]$ and $(k+15)(k-14) < 0$. This implies that $f(x) \geq f(k+1) \geq f(14) = 840$ for all $x \in [1, 14]$.
If $x \in [15, 20]$, then there exists $k \in \{15, 16, 17, 18, 19\}$ such that $x \in [k, k+1]$ and $(k+15)(k-14) > 0$. This implies that $f(x) \geq f(k) \geq f(15) = 840$ for all $x \in [15, 20]$. Finally, for $x \in [14, 15]$, we have that $k = 14$ and $f(x) = 840$ for all $x \in [14, 15]$. In conclusion, the minimum value of $f(x)$ is 840.

Answer: 840