EF EXAM
Texas A&M High School Math Contest
October 20, 2018

Directions: Answers should be simplified, and if units are involved include them in your answer.

1. Let N be the product of all numbers that appear on the 10×10 multiplication chart, i.e., the numbers of the form pq, where $1 \leq p, q \leq 10$. What is the largest number m such that $\sqrt[m]{N}$ is an integer?

2. How many real solutions does the following equation have?

$$(x+1)^{2018} + (x+1)^{2017}(x-2) + (x+1)^{2016}(x-2)^2 + \cdots + (x+1)(x-2)^{2017} + (x-2)^{2018} = 0$$

3. Let f be a continuous function on $[0, 2018]$ such that $f(x)f(2018-x) = 1$, for all $x \in [0, 2018]$. Evaluate

$$\int_0^{2018} \frac{dx}{1+f(x)}.$$

4. What is the number of natural numbers n with the property that $[\frac{n^2}{3}]$ is a prime number? Here, $[x]$ denotes the greatest integer that is not larger than x.

5. What is the coefficient of x^5 in the expansion of the following polynomial?

$$(1 + 2x + 3x^2 + 4x^3 + \cdots + 2018\,x^{2017})^2(1 + x^4 + x^8)^2$$

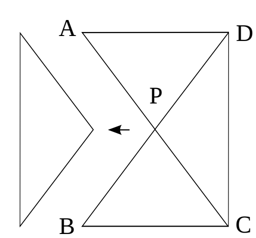
6. Consider the function $f(x, y) = y^2 - x^2 - 2xy + 2x + 1$. Jack and Janet play the following game: First, Jack plugs in a value for x, and then Janet plugs in a value for y. The value of the function will be considered as Jack's score. If Janet plays against Jack, what is the maximum score Jack can gain?

7. We say that a natural number greater than one has property S if the sum of any of its two distinct divisors is divisible by 7. How many numbers with property S are less than 100?

8. Consider the parabola $y = x^2 - 2ax + 1$ and the line $y = 2b(a - x)$. Let A be the set of points $(a, b) \in \mathbb{R}^2$ such that the line and the parabola defined above do not intersect. Find the area of A as a region in $\mathbb{R}^2$.

9. For any natural number n, let $p(n)$ be the product of the digits in the decimal expansion of n. Find $p(1) + p(2) + p(3) + \cdots + p(999)$.

10. Consider a rectangular paper $ABCD$ with $AB = 8$, $AD = 6$ and a point P, the intersection of the two diagonals. Remove the triangle $\triangle PAB$, and then fold PC and PD so that PA and PB are identified. Find the volume of the tetrahedron determined by the resulting piece of paper.



11. What is the maximum value of λ such that the following inequality holds for all $a > 0$?

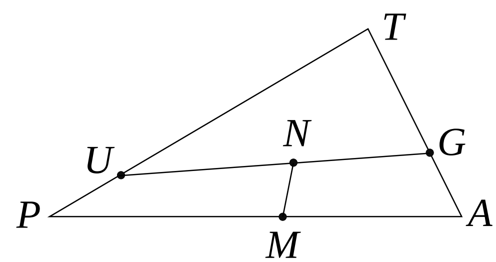
$$a^3 + \frac{1}{a^3} - 2 \geq \lambda(a + \frac{1}{a} - 2)$$

12. Suppose we have

$$\sum_{k=0}^{n-1} \sqrt[3]{\sqrt{ak^3 + bk^2 + ck + 1} - \sqrt{ak^3 + bk^2 + ck}} = \sqrt{n},$$

for all natural numbers n and some constants a, b, c. Find $a - b + c$.

13. In $\triangle PAT$, $\angle P = 36°$, $\angle A = 56°$, and $PA = 10$. Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1$. Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?



14. Evaluate the sum

$$\sum_{k=0}^{2017} (-1)^k \cos^{2018}\left(\frac{k\pi}{2018}\right).$$

15. Evaluate

$$\int_0^{\pi/3} \frac{dx}{5 + 4\cos(2x)}.$$

16. Evaluate the infinite series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right).$$

17. Consider the sequence

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

Determine $L = \lim_{n\to\infty} x_n$.

18. Evaluate the limit

$$\lim_{x\to\infty} \left(\sqrt{x} \cdot \int_x^{x+1} \sin(t^2)\, dt\right).$$

19. Evaluate the following sum.

$$1 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + \ldots$$