CD EXAM

Texas A&M High School Math Contest

November 9th, 2019

Directions: Use exact numbers. For example, if your answer includes π, e, square root etc, do not replace it by an approximate value.

1. A positive integer n written in base b is 25_b. If $2n$ is 52_b, what is b?

2. Given that 23^{100} is 137 digit number, find the number of digits of 23^{23}.

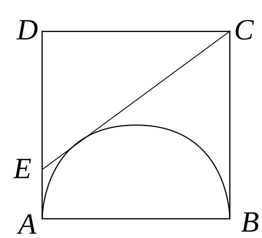
3. Let α and β be two solutions of $(x+2020)^2 - (x+2020) + 2019 = 0$. Find $(\alpha + 2019)(\beta + 2019)$.

4. Let P be the point $(3,1)$. Let Q be the reflection of P across the x- axis, let R be the reflection of Q about the line $y = x$ and let S be the reflection of R through the origin. What is the area of the quadrilateral $PQSR$?

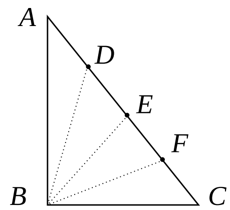
5. Assume that clock hands move continuously on the clock. Find the first (earliest) time and the last time when two hands overlap strictly between 12:00 AM and 12:00 PM. Write the answer as pairs (x, y), where x is hours and y is minutes.

6. Let P be a point on the circle $x^2 + y^2 = 9$. Find the length of locus of the centroid of $\triangle PQR$ where $Q = (2,5)$ and $R = (7,4)$.

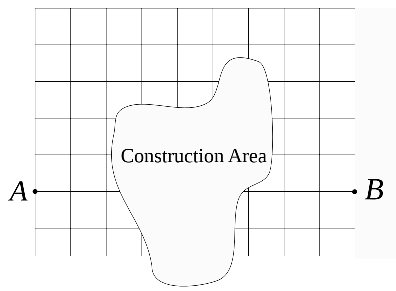
7. Square $ABCD$ has side length 2. A semicircle with diameter AB is constructed inside the square, and the tangent to semicircle from C intersects side AD at E. What is the length of CE?



8. Consider a triangle $\triangle ABC$ with $\angle B = 90°$. Suppose the distances from B to the quadrisection points D, E and F of $\overline{AC}$ are $\cos x$, x and $\sin x$ respectively. Find x.



9. The following map shows traffic system for two places A and B. Every square has side that equals 1 mile. Each car travels along horizontal and vertical grid lines. Find the number of shortest paths from A to B if one cannot cross the construction area.



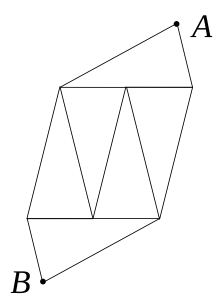
10. Solve the equation $4 \cdot 9^{x-1} = 3\sqrt{2^{2x+1}}$.

11. The line $y = k$, $-1 < k < 0$, intersects two graphs $y = \sin x$ and $y = \cos x$ at four points ($0 \le x < 2\pi$). Let a, b, c and d be the x-coordinates of the intersections. Find

$$\sin\left(\frac{a+b+c+d}{4}\right) + \cos\left(\frac{a+b+c+d}{4}\right) + \tan\left(\frac{a+b+c+d}{4}\right)$$

12. Find the number of subsets of $\{1, 2, 3, \cdots, 8\}$ that contain at least four consecutive numbers.

13. In the figure below, there are six non-overlapping congruent isosceles triangles. The sides of each triangle are 2, 2 and 1. Find the distance from A to B.



14. Let $X = \{1, 2, \cdots, 10\}$. Find the number of one-to-one functions f with domain X and range X such that x and $f(x)$ are mutually prime for every x in X.

15. Find $a+b+c+d$ if a, b, c and d satisfy the following conditions.

A: $10 \leq a, b, c, d \leq 20$

B: $ab - cd = 28$

C: $ad - bc = 110$

16. Find the smallest number n such that the following statement is true. A collection of n points on the coordinate plane with integer coordinates contains a pair of points such that the trisection points of the line joining those two points have integer coordinates.

17. Ninety nine people $p_1, p_2, \cdots p_{99}$ shake hands with each other. It was observed that each person p_i shook hands with precisely i people for every i, $1 \leq i \leq 98$. Find the number of people that p_{99} shook hands.

18. How many possible distinct integer solutions (a, b, c) does the equation have?

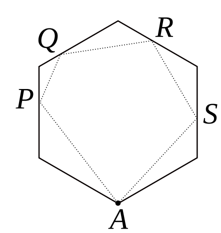
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = 1 \tag{1}$$

19. Let $x \neq 1$ be such that

$$\lfloor x \rfloor + \frac{2022}{\lfloor x \rfloor} = x^2 + \frac{2022}{x^2}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Find x^2.

20. Let A be a vertex of regular hexagon with side 1. Let P, Q, R and S be points on the four sides not containing A as in the figure. Find the minimum of $AP + PQ + QR + RS + SA$.



21. Find all integers $n \neq -1$ so that

$$\left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{2019}\right)^{2019}.$$