DE Exam
Texas A&M High School Math Contest
November, 2019

Answers should include units when appropriate.

1. The sides of a triangle have lengths of 15, 20, and 25. What is the length of the shortest altitude?

2. Increasing x by y percent gives 30; increasing y by x percent gives 25. Find x (assuming that it is positive).

3. Find the smallest n such that the product of the first n terms of the sequence $10^{1/11}, 10^{2/11}, 10^{3/11}, \ldots$ exceeds $100,000$.

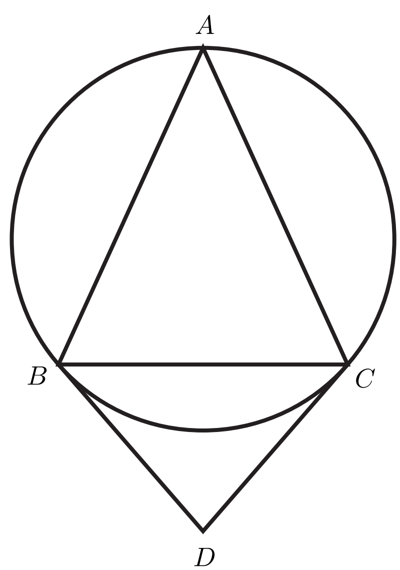
4. Suppose that r is a root of $x^4 + x^2 - 1$. Find $r^6 + 2r^4$.

5. All the students in an algebra class took a 100-point test. Five students scored 100, each student scored at least 60, and the mean score was 76. What is the smallest possible number of students in the class?

6. Solve the system

$$\begin{cases} 3^{\log_2 x} &= 4^{\log_2 y} \\ (4x)^{\log_2 4} &= (3y)^{\log_2 3}. \end{cases}$$

7. An acute isosceles triangle ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at D. If $\angle ABC = 2\angle CDB$, then find the radian measure of $\angle BAC$.



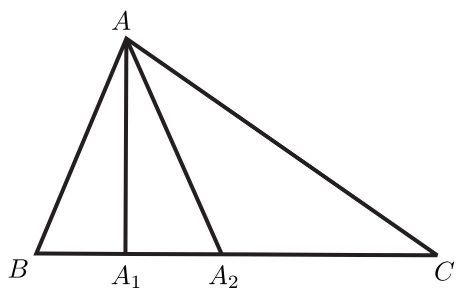
8. What is the exact value of $\frac{1}{2\sin 10^\circ} - 2\sin 70^\circ$ with no trigonometric functions in the answer?

9. Find the minimal value of $(x-1)(x-2)(x-3)(x-4)$.

10. Any five points are taken inside or on a square of side 1. Find the smallest possible number a such that it is always possible to select one pair of points from these five such that the distance between them is equal to or less than a.

11. Eight spheres of radius 1, one per octant, are each tangent to the coordinate planes of the octant. What is the radius of the smallest sphere, centered at the origin that contains these eight spheres?

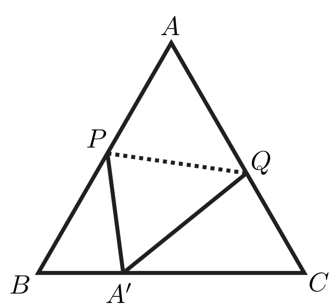
12. Let AA_1 be an altitude of triangle $\triangle ABC$, and let A_2 be the midpoint of the side BC. Suppose that AA_1 and AA_2 divide angle $\angle BAC$ into three equal angles. Find the angles of $\triangle ABC$.



13. Find all real solutions of the equation $\frac{8^x+2^x}{4^x-2} = 5$.

14. A subset of the set of integers $1, 2, 3, \ldots, 100$ has the property that none of its members is 3 times the other. What is the largest number of members such a subset can have?

15. Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on $\overline{BC}$ as shown. If $BA' = 1$ and $A'C = 2$, find the length of the crease PQ.



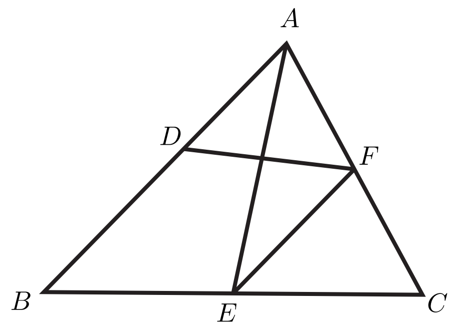
16. For which values of the parameter a does there exist exactly two pairs (x, y) of real numbers satisfying the system

$$\begin{cases} x = y^2 + a \\ y = x^2 + a, \end{cases}$$

17. How many solutions does the equation $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x$ have in the interval $[0, \pi]$.

18. Find a three-digit number equal to the sum of factorials of its digits. (Here factorial of n is the product of all integers from 1 to n.)

19. Triangle ABC has area 10. Points D, E, and F, all distinct from A, B, and C, are on sides AB, BC, and CA, respectively, and $AD = 2$, $DB = 3$. Triangle ABE and quadrilateral $DBEF$ have equal areas s. Find s.



20. Find all pairs (x, y) of numbers in the interval $[0, \pi/2]$ satisfying the system $\begin{cases} \sin^3 x = \frac{1}{2}\sin y \\ \cos^3 x = \frac{1}{2}\cos y. \end{cases}$

21. Simplify $\arctan \frac{1}{1+1+1^2} + \arctan \frac{1}{1+2+2^2} + \arctan \frac{1}{1+3+3^2} + \cdots + \arctan \frac{1}{1+n+n^2}$.