DE Exam
Texas A&M High School Math Contest
November, 2019

Answers should include units when appropriate.

1. Dividing the lengths by 5, we get $3, 4, 5$, hence it is a right triangle. The area of the original triangle is $15 \cdot 20/2 = 150$. The shortest altitude a will satisfy $25 \cdot a/2 = 150$, hence $a = 300/25 = 12$.

Answer: 12.

2. We have $x + \frac{xy}{100} = 30$ and $y + \frac{xy}{100} = 25$. Subtracting one equation from the other, we get $x - y = 5$. Substituting $y = x - 5$ into the first equation, we get $x + \frac{x(x-5)}{100} = 30$, i.e.,

$$x^2 + 95x - 3,000 = 0.$$

Therefore, $x = \frac{-95 \pm \sqrt{95^2 + 12,000}}{2} = \frac{-95 \pm 145}{2}$. The positive value is $x = 25$.

Answer: 25.

3. The product of the first n terms is equal to $10^{\frac{1+2+\cdots+n}{11}} = 10^{\frac{n(n+1)}{22}}$. It exceeds $100,000$ if and only if $\frac{n(n+1)}{22} > 5$, i.e., if $n(n+1) > 110$. We have $10 \cdot 11 = 110$, hence the answer is $n = 11$.

Answer: 11.

4. Divide the polynomial $r^6 + 2r^4$ by $r^4 + r^2 - 1$ using, for example, long division. We get $r^6 + 2r^4 = (r^4 + r^2 - 1)(r^2 + 1) + 1$, hence the answer is 1.

Answer: 1.

5. Let $60 + x_1, 60 + x_2, \ldots, 60 + x_n$ be the scores of the students, excluding the five students that scored 100. Then $0 \le x_i \le 40$, the number of students is $n + 5$, and we have

$$500 + 60n + x_1 + x_2 + \cdots + x_n = 76(n + 5).$$

Equivalently,

$$x_1 + x_2 + \cdots + x_n = 16n - 120.$$

Since $x_i \ge 0$, we have $0 \le 16n - 120$, hence $n \ge \frac{120}{16} = 7.5$. It follows that $n \ge 8$. On the other hand, we can have, for example, $x_1 = 8$ and $x_2 = x_3 = \cdots = x_8 = 0$.

Answer: 13 students.

6. Taking logarithms of the equations, we get

$$\begin{cases} \log_2 3 \log_2 x &= \log_2 4 \log_2 y \\ \log_2 4(\log_2 4 + \log_2 x) &= \log_2 3(\log_2 3 + \log_2 y). \end{cases}$$

The second equation is equivalent to

$$(\log_2 4)^2 + \log_2 4 \log_2 x = (\log_2 3)^2 + \log_2 3 \log_2 y.$$

Substituting $\log_2 y = \frac{\log_2 3}{\log_2 4} \log_2 x$, we get

$$(\log_2 4)^3 + (\log_2 4)^2 \log_2 x = \log_2 4 (\log_2 3)^2 + (\log_2 3)^2 \log_2 x,$$

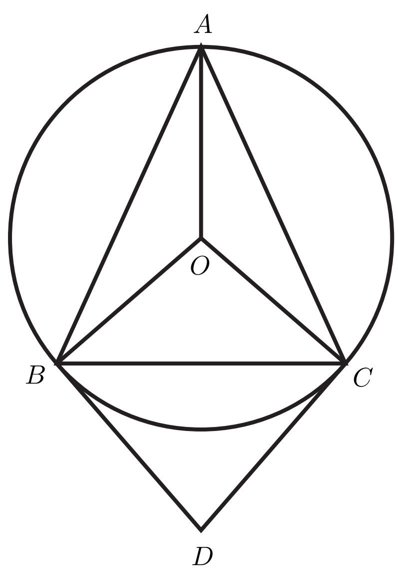
or

$$((\log_2 4)^2 - (\log_2 3)^2) \log_2 x = \log_2 4((\log_2 3)^2 - (\log_2 4)^2).$$

Dividing both sides by $(\log_2 4)^2 - (\log_2 3)^2$, we get $\log_2 x = -\log_2 4$, i.e., $x = 1/4$. Then $\log_2 y = -\log_2 3$, hence $y = 1/3$.

Answer: $x = 1/4, y = 1/3$.

7. Let O be the center of the circle, and let $\alpha = \angle BDC$. Since $\angle OBD$ and $\angle OCD$ are right angles, we have $\angle BOC + \angle BDC = \pi$, so $\angle BOC = \pi - \alpha$. Then $\angle BAC = \frac{\pi - \alpha}{2}$ as an inscribed angle. Hence $\angle ABC = \frac{1}{2}\left(\pi - \frac{\pi-\alpha}{2}\right) = \frac{\pi+\alpha}{4}$. On the other hand, by assumptions, $\angle ABC = 2\alpha$. Solving equation $2\alpha = \frac{\pi+\alpha}{4}$ we get $\alpha = \frac{\pi}{7}$. It follows that $\angle BAC = \frac{3\pi}{7}$.



Answer: $\frac{3\pi}{7}$.

8. Using the formula $\sin\alpha \sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$, we get $\frac{1}{2\sin 10^\circ} - 2\sin 70^\circ = \frac{1 - 4\sin 10^\circ \sin 70^\circ}{2 \sin 10^\circ} = \frac{1 - 2(\cos 60^\circ - \cos 80^\circ)}{2\sin 10^\circ} = \frac{1 - 1 + 2\cos 80^\circ}{2 \sin 10^\circ} = \frac{2\sin 10^\circ}{2\sin 10^\circ} = 1$.

Answer: 1.

9. We have $(x-1)(x-2)(x-3)(x-4) = (x-1)(x-4)\cdot(x-2)(x-3) = (x^2-5x+4)(x^2-5x+6)$. If we denote $t = x^2 - 5x + 4$, then our function is $t(t+2) = (t+1)^2 - 1$. The minimal value of $(t+1)^2 - 1$ is at $t = -1$ and is equal to -1. The value $-1 = x^2 - 5x + 4$ is attained, since the equation $x^2 - 5x + 5 = 0$ has real solutions (namely, $x = (5 \pm \sqrt{5})/2$).

Answer: -1

10. Divide the square into four squares of side $1/2$. Then out of any five points of the big square there will exist at least two points in one of the four little squares. The distance between them will be not more than the length of the diagonal of the little square, which is $\sqrt{2}/2$.

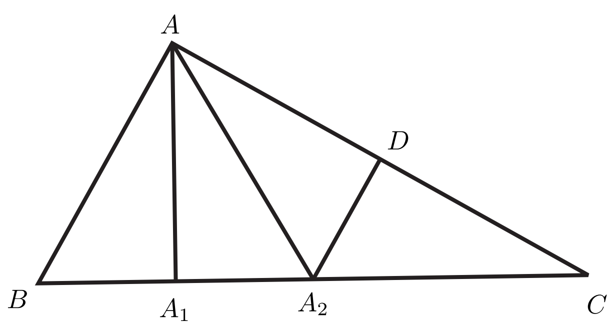
On the other hand, if we select the vertices of the big square and the central point, then the distance between any two selected points is at least $\sqrt{2}/2$.

Answer: $\sqrt{2}/2 = 1/\sqrt{2}$.

11. Since the distances from the center of every sphere to the planes of the octant are equal to 1, the centers have coordinates $(\pm 1, \pm 1, \pm 1)$. Then the distance from the origin to each center is equal to $\sqrt{3}$. It follows that the radius of the sphere containing all eight spheres is $\sqrt{3} + 1$.

Answer: $\sqrt{3} + 1$.

12. Triangles ABA_1 and AA_2A_1 are congruent, since they have a common side and equal angles adjacent to it. Let A_2D be the altitude of $\triangle AA_2C$. Then triangles AA_1A_2 and AA_2D are also congruent. It follows that the A_2D is twice shorter than A_2C. This implies that $\angle DCA_2 = \pi/6$. From right triangle AA_1C we conclude that $\angle A_1AC = \pi/3$, hence $\angle BAC = \pi/2$.



Answer: Angles of the triangle are $\pi/6, \pi/3, \pi/2$. Or $30°, 60°, 90°$.

13. Let $y = 2^x$. Then the equation can be written as $\frac{y^3+y}{y^2-2} = 5$, or $y^3 - 5y^2 + y + 10 = 0$. Note that $x = 1$ is a solution of the original equation, hence $y = 2$ is a solution of the second transformed equation. Dividing the polynomial by $(y-2)$, we get $y^2 - 3y - 5 = 0$. Solutions of this equation are $y = \frac{3\pm\sqrt{29}}{2}$. Since $\sqrt{29} > 3$ and $y > 0$, only $y = \frac{3+\sqrt{29}}{2}$ works, and we get $x = \log_2 \frac{3+\sqrt{29}}{2}$.

Answer: 1 and $\log_2 \frac{3+\sqrt{29}}{2}$.

14. We have the following chains of the form $(x, 3x, 9x, \ldots)$ in the set $\{1, 2, \ldots, 100\}$:

$(1, 3, 9, 27, 81), (2, 6, 18, 54), (4, 12, 36), (5, 15, 45), (7, 21, 63), (8, 24, 72), (10, 30, 90), (11, 33, 99),$

then chains of length two:

$$(13, 39), (14, 42), \ldots, (32, 96),$$

and single numbers not included in any chain:

$$34, 35, 37, \ldots, 98, 100.$$

Each chain starts with a number not divisible by 3, so that there are 20-6=14 chains of length 2 and 67-22=45 single numbers.

We can choose at most 3 numbers from $(1, 3, 9, 27, 81)$, and most 2 numbers from $(2, 6, 18, 54)$, at most 2 from each of the 6 chains of length 3, at most one from each chain of length 2, and we can choose each of the isolated points. In total we get that the largest number of members in a subset is

$$3 + 2 + 12 + 14 + 45 = 76.$$

Answer: 76.

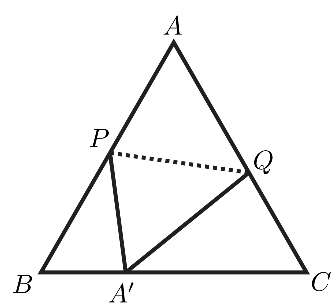
15. Let $a = BP$. Then by Theorem of Cosines: $(PA')^2 = a^2 + 1 - a$. On the other hand, $PA' = PA$, hence $PA' = 3 - a$. We get at equation $(3-a)^2 = a^2+1-a$, i.e., $9-6a = 1-a$, or $8 = 5a$ and $a = 8/5$. It follows that $PA = 3 - 8/5 = 7/5$.

Similarly, if $b = CQ$, then we have $(QA')^2 = b^2 + 4 - 2b$ and $QA' = QA = 3 - b$, hence $(3-b)^2 = b^2+4-2b$, or $9-6b = 4-2b$, so that $b = 5/4$. It follows that $QA = 3-5/4 = 7/4$.

Applying Theorem of Cosines to $\triangle APQ$ we get

$$PQ^2 = AP^2 + AQ^2 - AP \cdot AQ = \frac{49}{25} + \frac{49}{16} - \frac{49}{20} = \frac{49 \cdot 21}{400},$$

hence $PQ = \frac{7\sqrt{21}}{20}$.



Answer: $\frac{7\sqrt{21}}{20}$.

16. Subtracting one equality from the other, we get $x - y = y^2 - x^2$, which can be written as $(x-y)(1+x+y) = 0$. Consequently, either $x = y$, or $y = -1 - x$. In the first case our system becomes one equation $x^2 - x + a = 0$. Its discriminant is $1 - 4a$. Consequently, we get 0 solutions when $a > 1/4$, one if $a = 1/4$, and two if $a < 1/4$.

If $y = -1 - x$, then the system becomes $-1 - x = x^2 + a$, i.e., $x^2 + x + a + 1 = 0$. Its discriminant is $1 - 4(a+1) = -3 - 4a$. Consequently, we get this time 0 solutions when $a > -3/4$, one if $a = -3/4$, and two if $a < -3/4$.

Putting two cases together, we get that the system has 0 solutions if $a > 1/4$, 1 if $a = 1/4$, 2 if $-3/4 < a < 1/4$, 3 if $a = -3/4$, and 4 if $a < -3/4$.

Answer: for a in the interval $(-3/4, 1/4)$.

17. The left-hand side of the equation is $\sin x + \sin 2x + \sin 3x = \sin x + 2\sin x\cos x + \sin x\cos 2x + \sin 2x\cos x = \sin x + 2\sin x\cos x + \sin x\cos 2x + 2\sin x\cos^2 x = \sin x(1 + 2\cos x + \cos 2x + 2\cos^2 x) = \sin x(2 + 2\cos x + 2\cos 2x)$. Consequently, the equation is equivalent to

$$(1 + \cos x + \cos 2x)(2\sin x - 1) = 0.$$

We get $\sin x = 1/2$ or $1 + \cos x + \cos 2x = 0$. In the first case $x = \pi/6$ or $x = 5\pi/6$.

The second case is equivalent to $\cos x + 2\cos^2 x = 0$, i.e., to $\cos x = 0$ or $\cos x = -1/2$, which gives solutions $x = \pi/2$ or $x = 2\pi/3$.

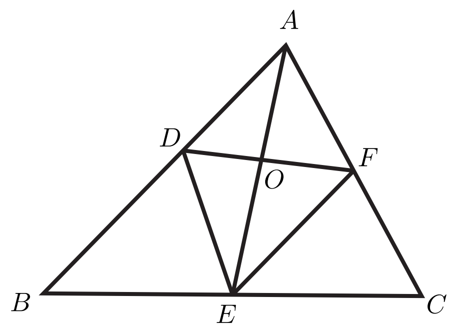
Answer: 4 solutions.

18. The number can use only digits $0, 1, 2, 3, 4, 5, 6$, since factorials of other digits have more than 3 digits. It follows that 6 is also not possible, since $6! > 666$. One of the digits must be then 5, otherwise our number is not greater than $3 \times 4! = 72 < 100$. Then the minimal digit must be not greater than 3, because our number is not greater than $3 \times 5! = 375 < 400$. It follows that the minimal digit is not greater than 2, because our number is not greater than $2 \times 5! + 2! = 244 < 300$. The middle (by size) digit must be not greater than 4 because $244 = 2 \times 5! + 2!$ and $241 = 2 \times 5! + 1$ do not satisfy the desired properties. Therefore the minimal digit must be 1, as our number is not greater than $5! + 4! + 2! = 148 < 200$. Let us inspect the remaining options:

- triple $(1, 4, 5) : 1! + 4! + 5! = 145$, and this number satisfies the properties;
- triple $(1, 3, 5) : 1! + 3! + 5! = 127$,no;
- triple $(1, 2, 5) : 1! + 2! + 5! = 123$,no;
- triple $(1, 1, 5) : 1! + 1! + 5! = 122$, no.

Answer: 145.

19. Denote by O the point of intersection of $\overline{DF}$ and $\overline{AE}$. Quadrilateral $BDOE$ is a common part of triangle ABE and quadrilateral $DBEF$. It follows that triangles ADO and EFO have equal area, which implies that $AO \cdot OD = FO \cdot OE$. Consequently, $AO : OE = FO : OD$, therefore triangles AFO and EOD are similar, and $\angle OAF = \angle OED$. This in turn implies that $\overline{AC}$ and $\overline{DE}$ are parallel. Consequently, $BE : BC = BD : BA = 3 : 5$. This implies that area of $\triangle ABE$ is $3/5$ times the are of $\triangle ABC$, i.e., that $s = 6$.



Answer: $s = 6$.

20. Square the equations and add them. We get $\sin^6 x + \cos^6 x = \frac{1}{4}(\sin^2 y + \cos^2 y) = \frac{1}{4}$. We have $\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) = \sin^4 x - \sin^2 x \cos^2 x + \cos^4 x = \sin^2 x(1 - \cos^2 x) - \sin^2 x \cos^2 x + \cos^2 x(1 - \sin^2 x) = 1 - 3\sin^2 x \cos^2 x$. It follows that $\sin^2 x \cos^2 x = \frac{1}{4}$. Consequently, $\sin^2 2x = 1$. Therefore, $\sin 2x = \pm 1$, and $x = \pi/4 + \pi n/2$ for some integer n. Since we are interested only in the values from the interval $[0, \pi/2]$, we have $x = \pi/4$. We have then $\sin x = \sqrt{2}/2$, hence $\sin y = \sqrt{2}/2$, and therefore also $y = \pi/4$.

Answer: $(x, y) = (\pi/4, \pi/4)$.

21. Let us prove by induction that the sum is equal to $\arctan \frac{n}{n+2}$. It is true for $n = 1$. We use the formula $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$. If the statement is true for $n - 1$, then the sum for n is equal to $\arctan \frac{n-1}{n+1} + \arctan \frac{1}{1+n+n^2}$, so its tangent is equal to

$$\frac{\frac{n-1}{n+1} + \frac{1}{1+n+n^2}}{1 - \frac{n-1}{(n+1)(1+n+n^2)}} = \frac{(n-1)(1+n+n^2) + n + 1}{(n+1)(1+n+n^2) - n + 1} = \frac{n^3 + n}{n^3 + 2n^2 + n + 2} = \frac{n(n^2+1)}{(n+2)(n^2+1)} = \frac{n}{n+2}.$$

Answer: $\arctan \frac{n}{n+2}$.