EF Exam Solutions
Texas A&M High School Math Contest
November 9, 2019

All answers must be simplified, and if units are involved, be sure to include them.

1. Find the minimum value of m for which the equation $x^2 + 2mx + 3m^2 + m - 21 = 0$ has real roots.

   Solution: For that to happen we need

$$\Delta = 4m^2 - 4(3m^2 + m - 21) \geq 0 \Leftrightarrow 2m^2 + m - 21 \leq 0 \Leftrightarrow m \in \left[-\frac{7}{2}, 3\right].$$

   So the minimum value is $-\frac{7}{2}$.

   Answer: $-\frac{7}{2}$

2. Let (x, y) be a solution of the system

$$\begin{cases} 25^x \cdot 125^y &= 1 \\ 1 \div 9^{3y} &= 81\sqrt{3}(\sqrt{3})^x. \end{cases}$$

   Find $x + y$.

   Solution: The first equation of the system is equivalent to

$$5^{2x} \cdot 5^{3y} = 1 \Leftrightarrow 2x + 3y = 0,$$

   and the second equation is equivalent to

$$\left(\frac{1}{3}\right)^{6y} = 3^4 \cdot 3^{1/2} \cdot 3^{x/2} \Leftrightarrow -6y = \frac{9+x}{2} \Leftrightarrow x + 12y = -9.$$

   Solving the system of equations $2x + 3y = 0$ and $x + 12y = -9$, we get $x = \frac{9}{7}$ and $y = -\frac{6}{7}$ which implies that $x + y = \frac{3}{7}$.

   Answer: $\frac{3}{7}$

3. Let a and b be the solutions of the equation $x^2 - 6x + 4 = 0$. Find the value of

$$(a^{2019} + b^{2019}) - 6(a^{2018} + b^{2018}) + 4(a^{2017} + b^{2017}) + a^2 + b^2.$$

   Solution: The above expression is equivalent to

$$a^{2017}(a^2 - 6a + 4) + b^{2017}(b^2 - 6b + 4) + (a + b)^2 - 2ab.$$

   Since a and b are solutions of the equation $x^2 - 6x + 4 = 0$ we get that $a^2 - 6a + 4 = 0$, $b^2 - 6b + 4 = 0$, $a + b = 6$, and $ab = 4$. Therefore, the value of the above expression is $0 + 0 + 6^2 - 2 \cdot 4 = 28$.

   Answer: 28

4. Let $f(x) = 4x^3 - 5x^2 + px + q$, where p and q are integers and suppose that $x^2 + 3x - 4$ is a factor of $f(x)$. Find pq.

Solution: Since $x^2 + 3x - 4 = (x-1)(x+4)$ we need $f(1) = 0$ and $f(-4) = 0$. $f(1) = 0$ is equivalent to $p + q = 1$ while $f(-4) = 0$ is equivalent to $-4p + q = 336$. Solving the system formed by the above two equations we get $p = -67$ and $q = 68$. This gives us that $pq = -4556$.

Answer: -4556

5. In the expansion of $(1 + ax - x^2)^8$ where a is a positive constant, the coefficient of x^2 is 244. Find the value of a.

Solution: Using the Binomial Theorem with $n = 8$ and $n = 7$ we can write

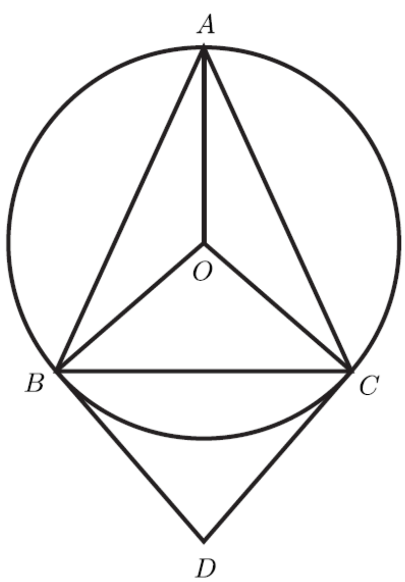
$$\begin{aligned}((1+ax) - x^2)^8 = (1+ax)^8 - 8(1+ax)^7x^2 + \ldots &= 1 + 8(ax) + 28(ax)^2 + \ldots \\ &- 8(1 + 7(ax) + \ldots)x^2,\end{aligned}$$

where dots denote terms that do not affect the coefficient of x^2. The coefficient of x^2 is $28a^2 - 8$. We solve the equation $28a^2 - 8 = 244$ for positive a and we get $a = 3$.

Answer: 3

6. An acute isosceles triangle ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at D. If $\angle ABC = 2\angle CDB$, then find the radian measure of $\angle BAC$.

Solution:



Let O be the center of the circle, and let $\alpha = \angle BDC$. Since $\angle OBD$ and $\angle OCD$ are right angles, we have $\angle BOC + \angle BDC = \pi$, so $\angle BOC = \pi - \alpha$. Then $\angle BAC = \dfrac{\pi - \alpha}{2}$ as an inscribed angle. Hence $\angle ABC = \dfrac{1}{2}\left(\pi - \dfrac{\pi - \alpha}{2}\right) = \dfrac{\pi + \alpha}{4}$. On the other hand, by assumptions, $\angle ABC = 2\alpha$. Solving equation $2\alpha = \dfrac{\pi + \alpha}{4}$ we get $\alpha = \dfrac{\pi}{7}$. It follows that $\angle BAC = \dfrac{3\pi}{7}$.

Answer: $\dfrac{3\pi}{7}$

7. Let $P(x) = (5x^3 + 2x^2 - 4x + 6)^4$. If $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, find $a_1 + a_3 + a_5 + \cdots$.

Solution: The sum of the coefficients of the x^n terms in the expansion of $P(x)$, where n is an odd number is given by $\frac{1}{2}(P(1) - P(-1))$. We see that $P(1) = (5 + 2 - 4 + 6)^4 = 9^4 = 6561$ and $P(-1) = (-5 + 2 + 4 + 6)^4 = 7^4 = 2401$ and that $\frac{1}{2}(P(1) - P(-1)) = 2080$.

Answer: 2080

8. Find $\lim_{n\to\infty} x_n$ where

$$x_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right), \quad n \geq 2.$$

Solution: We can write x_n as

$$\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n-1}\right)\left(1 + \frac{1}{n-1}\right)\left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right).$$

We notice that $\left(1 + \frac{1}{k-1}\right)\left(1 - \frac{1}{k}\right) = 1$ for all $k = 3, 4, \ldots n$ so we get that $x_n = \frac{n+1}{2n}$, which implies that $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Answer: $\frac{1}{2}$

9. Let $f(x) = 4x^m + 5x^{-n}$, where m and n are positive integers. If $x^2f''(x) + 2xf'(x) = 6f(x)$ find $m + n$.

Solution: We have that

$$f'(x) = 4mx^{m-1} - 5nx^{-n-1}$$

and that

$$f''(x) = 4m(m-1)x^{m-2} + 5n(n+1)x^{-n-2}.$$

The above equation becomes

$$4m(m-1)x^m + 5n(n-1)x^{-n} + 8mx^m - 10nx^{-n} = 24x^m + 30x^{-n},$$

which gives rise to the system of equations $m^2 + m - 6 = 0$ and $n^2 - n - 6 = 0$. Since m and n are positive integers we get that $m = 2$ and $n = 3$ which implies that $m + n = 5$.

Answer: 5

10. Solve the equation

$$(x+1)^{\log_3(x-2)} + 2(x-2)^{\log_3(x+1)} = 3x^2 + 6x + 3.$$

Solution: The solution(s) must satisfy the conditions $x - 2 > 0$ and $x + 1 > 0$. We notice that $a^{\log_c b} = b^{\log_c a}$, for any $a, b \in (0, \infty)$ and $c \in (0, \infty) \setminus \{1\}$. Using this fact our equation can be written as

$$(x+1)^{\log_3(x-2)} + 2(x+1)^{\log_3(x-2)} = 3(x+1)^2 \Leftrightarrow 3(x+1)^{\log_3(x-2)} = 3(x+1)^2$$
$$\Leftrightarrow \log_3(x-2) = 2 \Leftrightarrow x - 2 = 3^2 \Leftrightarrow x = 11.$$

$x = 11$ is a solution of the equation since it satisfies the conditions.

Answer: 11

11. Any five points are taken inside or on a square of side 1. Find the smallest possible number a such that it is always possible to select one pair of points from these five such that the distance between them is equal to or less than a.

Solution: Divide the square into four squares of side $\frac{1}{2}$. Then out of any five points of the big square there will exists at least two points in one of the four little squares. The distance between them will be not more than the length of the diagonal of the little square, which is $\frac{\sqrt{2}}{2}$.
On the other hand, if we select the vertices of the big square and the central point, then the distance between any two selected points is at least $\frac{\sqrt{2}}{2}$.

Answer: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

12. Find the product of all the solutions in $[0, 2\pi)$ of the inequality

$$\sin 4x - \sqrt{2}\cos\left(4x - \frac{\pi}{4}\right) \geq 1.$$

Solution. The left side of the inequality can be written as

$$\sin 4x - \sqrt{2}\left(\cos 4x \cos\frac{\pi}{4} + \sin 4x \sin\frac{\pi}{4}\right) = \sin 4x - \sqrt{2}\left(\frac{\sqrt{2}}{2}\cos 4x + \frac{\sqrt{2}}{2}\sin 4x\right) = -\cos 4x.$$

Then our inequality becomes

$$-\cos 4x \geq 1 \Leftrightarrow \cos 4x \leq -1 \Leftrightarrow \cos 4x = -1 \Leftrightarrow 4x = \pi + 2k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}.$$

The solutions in $[0, 2\pi)$ are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. Therefore, their product is $\frac{105\pi^4}{256}$.

Answer: $\frac{105\pi^4}{256}$

13. Find the exact value of the integral $\int_0^1 x\ln(x+1)dx$.

Solution: First we integrate by parts to get that

$$\int_0^1 x\ln(x+1)dx = \int_0^1 \ln(x+1)\left(\frac{x^2}{2}\right)' dx = \frac{x^2}{2}\ln(x+1)\Big|_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{1}{x+1}dx = \frac{1}{2}\ln 2 - \frac{1}{2}\int_0^1 \frac{x^2}{x+1}dx.$$

Second we find that

$$\begin{aligned}\int_0^1 \frac{x^2}{x+1}dx &= \int_0^1 \frac{x^2-1}{x+1}dx + \int_0^1 \frac{1}{x+1}dx = \int_0^1 (x-1)dx + \int_0^1 \frac{1}{x+1}dx \\ &= \left(\frac{x^2}{2} - x + \ln(x+1)\right)\Bigg|_0^1 = \ln 2 - \frac{1}{2}.\end{aligned}$$

So $\int_0^1 x\ln(x+1)dx = \frac{1}{2}\ln 2 - \frac{1}{2}\left(\ln 2 - \frac{1}{2}\right) = \frac{1}{4}$.

Answer: $\frac{1}{4}$

14. It is given that $\sin\theta + \cos\theta = \sqrt[4]{3}$. Find the exact value of $\sin^5\theta + \cos^5\theta$.

Solution: Denote $\sin\theta + \cos\theta = k$ and use the formula $a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$ to write

$$\sin^5\theta + \cos^5\theta = k[\sin^4\theta + \sin^2\theta\cos^2\theta + \cos^4\theta - \sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)]$$

Using the fact that $\sin^2\theta + \cos^2\theta = 1$ we obtain that

$$\sin^5\theta + \cos^5\theta = k[(\sin^2\theta + \cos^2\theta)^2 - \sin^2\theta\cos^2\theta - \sin\theta\cos\theta] = k(1 - \sin\theta\cos\theta - \sin^2\theta\cos^2\theta).$$

Since $2\sin\theta\cos\theta = (\sin\theta + \cos\theta)^2 - 1 = k^2 - 1$ we get that

$$\sin^5\theta + \cos^5\theta = k\left[1 - \frac{k^2-1}{2} - \frac{(k^2-1)^2}{4}\right] = \frac{-k^5 + 5k}{4}.$$

In our problem $k = \sqrt[4]{3}$ which gives us $\sin^5\theta + \cos^5\theta = \dfrac{\sqrt[4]{3}}{2}$.

Answer: $\dfrac{\sqrt[4]{3}}{2}$

15. Find the exact value of the expression

$$\sin 1^\circ \left(\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 59^\circ \cos 60^\circ}\right).$$

Solution: We notice that for any integer n from 0 to 59 we have

$$\tan(n+1)^\circ - \tan n^\circ = \frac{\sin(n+1)^\circ \cos n^\circ - \cos(n+1)^\circ \sin n^\circ}{\cos(n+1)^\circ \cos n^\circ} = \frac{\sin 1^\circ}{\cos n^\circ \cos(n+1)^\circ}.$$

So our expression is equal to

$$(\tan 1^\circ - \tan 0^\circ) + (\tan 2^\circ - \tan 1^\circ) + \cdots + (\tan 60^\circ - \tan 59^\circ) = \tan 60^\circ - \tan 0^\circ = \sqrt{3}.$$

Answer: $\sqrt{3}$

16. Find the largest real solution of the equation

$$(x-1)(x-3)(x-5)(x-7)(x-9)(x-11) = -225.$$

Solution: If we denote $(x-6)^2 = y$ then $(x-5)(x-7) = y-1$, $(x-3)(x-9) = y-9$, and $(x-1)(x-11) = y-25$. Our equation becomes

$$(y-1)(y-9)(y-25) = -225 \Leftrightarrow y^3 - 35y^2 + 259y = 0 \Leftrightarrow y(y^2 - 35y + 259) = 0.$$

By using the quadratic formula we get that the solutions of $y^2 - 35y + 259 = 0$ are $y = \dfrac{35 - 3\sqrt{21}}{2}$ and $y = \dfrac{35 + 3\sqrt{21}}{2}$. Therefore, the largest solution of our initial equation is $x = 6 + \sqrt{\dfrac{35 + 3\sqrt{21}}{2}}$.

Answer: $6 + \sqrt{\dfrac{35 + 3\sqrt{21}}{2}} = 6 + \dfrac{\sqrt{70 + 6\sqrt{21}}}{2}$

17. Evaluate the integral $\int_0^{\frac{\pi}{2}} \sin^8 x dx$. (Hint: Differentiate the function $\sin^{n-1} x \cos x$.)

Solution: For every positive integer n with $n \geq 2$ we have due to the product rule that

$$\begin{aligned}\frac{d}{dx}(\sin^{n-1} x \cos x) &= (n-1)\sin^{n-2} x \cos^2 x - \sin^n x = (n-1)\sin^{n-2} x(1-\sin^2 x) - \sin^n x \\ &= -n\sin^n x + (n-1)\sin^{n-2} x.\end{aligned}$$

Integrating the equation from above for $n = 8$ we get

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \frac{d}{dx}(\sin^7 x \cos x)dx = -8\int_0^{\frac{\pi}{2}} \sin^8 x dx + 7\int_0^{\frac{\pi}{2}} \sin^6 x dx \Leftrightarrow \\ &\sin^7 x \cos x\Big|_0^{\frac{\pi}{2}} = -8\int_0^{\frac{\pi}{2}} \sin^8 x dx + 7\int_0^{\frac{\pi}{2}} \sin^6 x dx \Leftrightarrow \\ &0 = -8\int_0^{\frac{\pi}{2}} \sin^8 x dx + 7\int_0^{\frac{\pi}{2}} \sin^6 x dx \Leftrightarrow \int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8}\int_0^{\frac{\pi}{2}} \sin^6 x dx.\end{aligned}$$

We can repeat the process to obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^8 x dx &= \frac{7}{8}\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{7}{8}\cdot\frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x dx \\ &= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} 1 dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}.\end{aligned}$$

Answer: $\frac{35\pi}{256}$

18. Consider the sequence $(a_n)_{n\geq 1}$, with

$$a_n = \lim_{x\to 0}(1 - x\sin nx)^{1/x^2}.$$

Find $\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n)$.

Solution: We have that

$$a_n = \lim_{x\to 0}\left[(1 - x\sin nx)^{\frac{1}{-x\sin nx}}\right]^{-\frac{x\sin nx}{x^2}} = e^{-n\lim\limits_{x\to 0}\dfrac{\sin nx}{nx}} = e^{-n},$$

since $\lim_{x\to 0}(1+x)^{1/x} = e$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Therefore,

$$a_1 + a_2 + \cdots + a_n = e^{-1} + e^{-2} + \cdots + e^{-n} = e^{-1}\cdot\frac{e^{-n} - 1}{e^{-1} - 1}.$$

Using the fact that $\lim_{n\to\infty} e^{-n} = 0$ we deduce that

$$\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = \frac{-e^{-1}}{e^{-1} - 1} = \frac{1}{e - 1}.$$

Answer: $\frac{1}{e-1}$

19. Simplify $\arctan \frac{1}{1+1+1^2} + \arctan \frac{1}{1+2+2^2} + \arctan \frac{1}{1+3+3^2} + \cdots + \arctan \frac{1}{1+n+n^2}$.

Solution: Let us prove by induction that the sum is equal to $\arctan \dfrac{n}{n+2}$. It is true for $n = 1$. We use the formula $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$. If the statement is true for $n - 1$, then the sum for n is equal to $\arctan \dfrac{n-1}{n+1} + \arctan \dfrac{1}{1+n+n^2}$, so its tangent is equal to

$$\frac{\dfrac{n-1}{n+1} + \dfrac{1}{1+n+n^2}}{1 - \dfrac{n-1}{(n+1)(1+n+n^2)}} = \frac{(n-1)(1+n+n^2)+n+1}{(n+1)(1+n+n^2)-n+1} = \frac{n^3+n}{n^3+2n^2+n+2} = \frac{n(n^2+1)}{(n+2)(n^2+1)} = \frac{n}{n+2}.$$

Answer: $\arctan \dfrac{n}{n+2}$

20. Find the value of the limit

$$L = \lim_{x\to\infty} \int_0^x \frac{1}{(1+t^2)(1+t^4)} dt.$$

Solution: By making the trigonometric substitution $t = \tan\theta$ we get

$$\int_0^x \frac{1}{(1+t^2)(1+t^4)} dt = \int_0^{\arctan x} \frac{\sec^2\theta}{(1+\tan^2\theta)(1+\tan^4\theta)} d\theta = \int_0^{\arctan x} \frac{1}{1+\tan^4\theta} d\theta.$$

Then our limit becomes

$$L = \lim_{x\to\infty} \int_0^{\arctan x} \frac{1}{1+\tan^4\theta} d\theta = \int_0^{\lim\limits_{x\to\infty} \arctan x} \frac{1}{1+\tan^4\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^4\theta} d\theta.$$

To find the last integral we make the substitution $\alpha = \dfrac{\pi}{2} - \theta$ and we get

$$L = -\int_{\frac{\pi}{2}}^0 \frac{1}{1+\cot^4\alpha} d\alpha = \int_0^{\frac{\pi}{2}} \frac{\tan^4\alpha}{1+\tan^4\alpha} d\alpha = \frac{\pi}{2} - L,$$

which implies that $L = \dfrac{\pi}{4}$.

Answer: $\dfrac{\pi}{4}$