EF Exam Solutions
Texas A&M High School Math Contest
November 4, 2023

All answers must be simplified, and if units are involved, be sure to include them.

1. Find $p+q$ if p and q are rational numbers such that $\sqrt{p+q\sqrt{7}} = \dfrac{9}{4-\sqrt{7}}$.

   Solution: We have

$$\sqrt{p+q\sqrt{7}} = \frac{9(4+\sqrt{7})}{(4-\sqrt{7})(4+\sqrt{7})} = \frac{9(4+\sqrt{7})}{16-7} = 4+\sqrt{7},$$

   which implies that $p+q\sqrt{7} = 23+8\sqrt{7}$. Since p and q are rational numbers and $\sqrt{7}$ is irrational, we get $p=23$, $q=8$ and $p+q=31$.

   Answer: 31

2. How many pairs of integers (x,y) are solutions of the equation $x^2-xy+y^2 = x+y$.

   Solution: Our equation can be written as

$$2x^2-2xy+2y^2-2x-2y+2 = 2 \Leftrightarrow (x-y)^2+(x-1)^2+(y-1)^2 = 2.$$

   Since x and y are integers, one of the three terms in the left-hand side of the last equation must be equal to zero, and the other two both equal to 1. Therefore, we have the following cases:

   (a) $x=y$. Then $|x-1| = |y-1| = 1$, so either $x=y=0$ or $x=y=2$. We get two solutions in this case.
   (b) $x=1$. Then $|y-1| = 1$ so either $y=0$ or $y=2$. We get two solutions in this case.
   (c) $y=1$. Then $|x-1| = 1$ so either $x=0$ or $x=2$. We get two solutions in this case.

   So, all together there are 6 solutions.

   Answer: 6

3. Let f be an increasing function and g be a function such that

$$f(g(x)+2023) \le f(x) \le (f\circ g)(x+2023),$$

   for all real numbers x. Find $g(0)$.

   Solution: Since f is increasing, from $f(g(x)+2023) \le f(x)$ we get $g(x)+2023 \le x \Leftrightarrow g(x) \le x-2023$ and from $f(x) \le (f(g(x+2023))$ we get $x \le g(x+2023)$ for all real numbers x. Replacing x with $x-2023$ in the last inequality, we obtain $x-2023 \le g(x)$. Therefore, $g(x) = x-2023$ which implies $g(0) = -2023$.

   Answer: -2023

4. Find the exact value of

$$\sqrt{36^{\log_6 5} + 10^{1-\log 2} - 3^{\log_9 36} + 1}.$$

   Solution: We use the identity $a^{\log_a x} = x$ to write

$$36^{\log_6 5} = (6^2)^{\log_6 5} = (6^{\log_6 5})^2 = 5^2 = 25, \quad 3^{\log_9 36} = (\sqrt{9})^{\log_9 36} = \sqrt{9^{\log_9 36}} = \sqrt{36} = 6.$$

   Since $1-\log 2 = \log 10 - \log 2 = \log\left(\frac{10}{2}\right) = \log 5$, we get that $10^{1-\log 2} = 10^{\log 5} = 5$. So the exact value of our given number is $\sqrt{25+5-6+1} = 5$.

   Answer: 5

5. When the polynomial $P(x) = 3x^4 + mx^3 + nx^2 + 2x - 15$ is divided by $3x^2 + x - 2$, the remainder is $x - 5$. Find $m^2 + n^2$.

Solution: There exists a polynomial $Q(x)$ such that

$$P(x) = (3x^2 + x - 2)Q(x) + x - 5 = (3x - 2)(x + 1)Q(x) + x - 5,$$

for all x. We get

$$3x^4 + mx^3 + nx^2 + 2x - 15 = (3x - 2)(x + 1)Q(x) + x - 5,$$

for all x. By substituting $x = -1$ in the above equation, we obtain $3 - m + n - 17 = -6$, which implies $-m + n = 8$. By substituting $x = \dfrac{2}{3}$ into the same equation we get $\frac{16}{27} + \frac{8}{27}m + \frac{4}{9}n + \frac{4}{3} - 15 = \frac{2}{3} - 5$, which after simplification leads to $2m + 3n = 59$. Solving the system consisting of these equations yields $m = 7$ and $n = 15$ which implies that $m^2 + n^2 = 274$.

Answer: 274

6. Consider the ellipse with vertices at $(0, -6)$ and $(0, 6)$ and passing through the point $(2, -4)$. Find the x-coordinate of the point where the ellipse intersects the positive x-axis.

Solution: The standard form of the equations of ellipses centered at (h, k) with vertical major axis is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

The center is the middle point of the line segment connecting the vertices so in this case it is the origin. The distance between the vertices is $2a$ so we get $2a = 12 \Rightarrow a = 6$. This implies that an equation of the ellipse is $\dfrac{x^2}{b^2} + \dfrac{y^2}{36} = 1$. Using the fact that the point $(2, -4)$ is on the ellipse we get that

$$\frac{4}{b^2} + \frac{16}{36} = 1 \Rightarrow b^2 = \frac{36}{5} \Rightarrow b = \frac{6}{\sqrt{5}}.$$

Answer: $\dfrac{6}{\sqrt{5}}$ or $\dfrac{6\sqrt{5}}{5}$

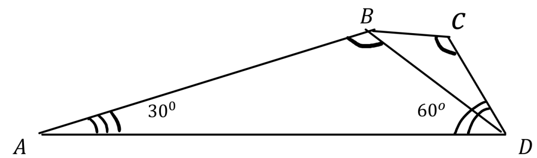
7. Find the sum of all distinct real solutions of the equation

$$(3x^2 - 4x + 1)^3 + (x^2 + 4x - 5)^3 = 64(x^2 - 1)^3.$$

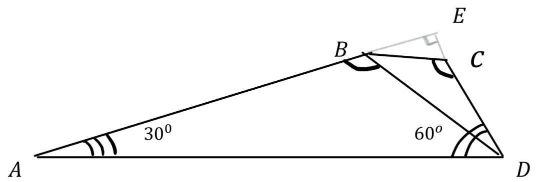
Solution: If we denote $a = 3x^2 - 4x + 1$ and $b = x^2 + 4x - 5$ then $a + b = 4x^2 - 4 = 4(x^2 - 1)$. So the above equation becomes $a^3 + b^3 = (a + b)^3 \Leftrightarrow 3ab(a + b) = 0 \Leftrightarrow a = 0$ or $b = 0$ or $a + b = 0$. This means $3x^2 - 4x + 1 = 0$ or $x^2 + 4x - 5 = 0$ or $x^2 - 1 = 0$. The three quadratic equations yield the different solutions $x = -5$, $x = -1$, $x = \dfrac{1}{3}$, and $x = 1$. Their sum is $-\dfrac{14}{3}$.

Answer: $-\dfrac{14}{3}$

8. Consider the quadrilateral $ABCD$ with $BD = 10$, $BC = 5$, $m\angle BAD = 30°$, $m\angle CDA = 60°$, and $m\angle ABD = m\angle BCD$. Find CD.



Solution: We extend the segments AB and DC until they intersect at the point E. By assumptions, the angle $m\angle AED = 90°$.



Since $m\angle BCE = m\angle DBE$ we get that $\triangle BCE \sim \triangle DBE$, so $\dfrac{CE}{BE} = \dfrac{BC}{DB} = \dfrac{5}{10}$ which implies that $BE = 2CE$. Applying the Pythagorean Theorem in the right triangle BCE we get $CE^2 + (2CE)^2 = 5^2 \Rightarrow CE = \sqrt{5}$ and $BE = 2\sqrt{5}$. Since $\dfrac{BE}{DE} = \dfrac{5}{10}$ as well, we have $DE = 2BE = 4\sqrt{5}$. Therefore, $CD = DE - CE = 4\sqrt{5} - \sqrt{5} = 3\sqrt{5}$.

Answer: $3\sqrt{5}$

9. Let f be a differentiable function such that $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 + 2h$ for all x and h and $f(0) = 1$. If $g(x) = e^{-x}f(x)$, find $g'(3)$.

Solution: We have

$$g'(x) = -e^{-x}f(x) + e^{-x}f'(x),$$
$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2 + 2) = 3x^2 + 2 \text{ and}$$
$$f(x) = x^3 + 2x + f(0) = x^3 + 2x + 1.$$

Therefore, $g'(3) = -34e^{-3} + 29e^{-3} = -5e^{-3}$.

Answer: $-5e^{-3}$ or $-\dfrac{5}{e^3}$

10. Suppose that the lengths of the sides of a triangle are three consecutive integers. Find the perimeter of the triangle if we know that the perimeter is (numerically) half of the area of the triangle.

Solution: Let $x-1$, x and $x+1$ be the lengths of the sides of the triangle, where x is an integer with $x > 2$. Using Heron's Formula we can write

$$A = \sqrt{\frac{3x}{2}\cdot\frac{x+2}{2}\cdot\frac{x}{2}\cdot\frac{x-2}{2}} = \frac{x}{4}\sqrt{3(x^2-4)}.$$

From the fact that $P = \dfrac{A}{2}$ we get

$$3x = \frac{x}{8}\sqrt{3(x^2-4)} \Rightarrow 192 = x^2 - 4 \Leftrightarrow x^2 = 196 \Rightarrow x = 14 \Rightarrow P = 42.$$

Answer: 42

11. Find the length of the graph of the function $f(x) = \left(\dfrac{x}{2}\right)^2 - \ln\sqrt{x}$ on the interval $[1, e]$.

Solution: The length is given by

$$l(f) = \int_1^e \sqrt{1 + (f'(x))^2}dx.$$

We differentiate $f(x)$ and we get

$$f'(x) = \frac{x}{2} - \frac{1}{2x} \Rightarrow 1 + (f'(x))^2 = \frac{1}{4}\left(x + \frac{1}{x}\right)^2.$$

Therefore,

$$l(f) = \frac{1}{2}\int_1^e \left(x + \frac{1}{x}\right) dx = \left(\frac{x^2}{4} + \frac{\ln x}{2}\right)\Bigg|_1^e = \frac{e^2+1}{4}.$$

Answer: $\frac{e^2+1}{4}$

12. Find the sum (in radians) of the solutions in $[0, 2\pi]$ of the equation

$$(\sqrt{3}+1)\cos x + (\sqrt{3}-1)\sin x = 2.$$

Solution: Note that $\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, which can be easily obtained from the fact that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and the standard formulas for $\cos\frac{x}{2}$ and $\sin\frac{x}{2}$ via $\cos x$. Then our equation can be written as

$$\frac{\sqrt{6}+\sqrt{2}}{4}\cos x + \frac{\sqrt{6}-\sqrt{2}}{4}\sin x = \frac{\sqrt{2}}{2} \Leftrightarrow \cos\frac{\pi}{12}\cos x + \sin\frac{\pi}{12}\sin x = \frac{\sqrt{2}}{2} \Leftrightarrow \cos\left(x - \frac{\pi}{12}\right) = \frac{\sqrt{2}}{2}.$$

Since $-\frac{\pi}{12} \le x - \frac{\pi}{12} \le \frac{23\pi}{12}$, we get that $x - \frac{\pi}{12} = \frac{\pi}{4}$ or $x - \frac{\pi}{12} = \frac{7\pi}{4}$. The solutions are $x_1 = \frac{\pi}{3}$ and $x_2 = \frac{11\pi}{6}$. Their sum is $x_1 + x_2 = \frac{13\pi}{6}$.

Answer: $\frac{13\pi}{6}$

13. Find the limit

$$L = \lim_{x\to\infty} \frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}}.$$

Solution: From the Fundamental Theorem of Calculus we know that

$$\frac{d}{dx}\int_0^x (\arctan t)^2 dt = (\arctan x)^2.$$

Applying L'Hospital's Rule we get

$$L = \lim_{x\to\infty} \frac{(\arctan x)^2}{\frac{x}{\sqrt{x^2+1}}} = \frac{\pi^2}{4}.$$

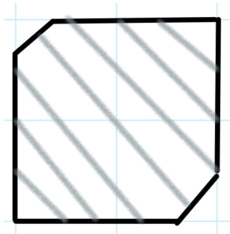
Answer: $\frac{\pi^2}{4}$

14. Point A is chosen at random from the line segment joining $(0,0)$ and $(2,0)$ as the center of a circle of radius 1. Point B is chosen at random from the line segment joining $(0,1)$ and $(2,1)$ as the center of another circle of radius 1. What is the probability that the two circles intersect?

Solution: Let the centers of the circles be $(a,0)$ and $(b,1)$. Then the circles intersect if and only if

$$\sqrt{(a-b)^2+1} \le 2 \Leftrightarrow (a-b)^2 \le 3 \Leftrightarrow -\sqrt{3} \le a - b \le \sqrt{3}.$$

The set of such (a, b) is the subset of the (a, b)-plane obtained by removing from the square $[0, 2] \times [0, 2]$ two right isosceles triangles with vertices $(2, 0)$ and $(0, 2)$ and legs of length $2 - \sqrt{3}$.



The area of this region is $4 - (2 - \sqrt{3})^2 = 4\sqrt{3} - 3$. Therefore, the probability is this area divided by the area of the square, or $\dfrac{4\sqrt{3} - 3}{4}$.

Answer: $\dfrac{4\sqrt{3} - 3}{4}$

15. Let x_1, x_2, and x_3 be the roots of the equation $x^3 - x + 1 = 0$. Find $x_1^{11} + x_2^{11} + x_3^{11}$.

Solution: We denote $s_n = x_1^n + x_2^n + x_3^n$ and we notice that s_n satisfies the recurrence relation $s_{n+3} = s_{n+1} - s_n$ since x_1, x_2, and x_3 are solutions of the equation $x^3 = x - 1$. Using Vieta's formulas we can write

$$\begin{aligned}
s_1 &= x_1 + x_2 + x_3 = 0, \\
s_2 &= x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1) = 2, \\
s_3 &= x_1^3 + x_2^3 + x_3^3 = x_1 + x_2 + x_3 - 3 = -3.
\end{aligned}$$

Using the recurrence relation we get $s_4 = s_2 - s_1 = 2$, $s_5 = s_3 - s_2 = -5$, $s_6 = s_4 - s_3 = 5$, $s_7 = s_5 - s_4 = -7$, $s_8 = s_6 - s_5 = 10$, $s_9 = s_7 - s_6 = -12$ and $s_{11} = s_9 - s_8 = -22$.

Answer: -22

16. Find $\theta \in (0, 180^\circ)$ such that

$$\cot\theta = \frac{(1 + \tan 1^\circ)(1 + \tan 2^\circ) - 2}{(1 - \tan 1^\circ)(1 - \tan 2^\circ) - 2}.$$

Solution: We are using the formulas

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \quad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

to write

$$\begin{aligned}
\cot\theta &= \frac{\tan 1^\circ + \tan 2^\circ + \tan 1^\circ \tan 2^\circ - 1}{-\tan 1^\circ - \tan 2^\circ + \tan 1^\circ \tan 2^\circ - 1} = \frac{(\tan 1^\circ + \tan 2^\circ) - (1 - \tan 1^\circ \tan 2^\circ)}{-(\tan 1^\circ + \tan 2^\circ) - (1 - \tan 1^\circ \tan 2^\circ)} \\
&= -\frac{\frac{\tan 1^\circ + \tan 2^\circ}{1 - \tan 1^\circ \tan 2^\circ} - 1}{\frac{\tan 1^\circ + \tan 2^\circ}{1 - \tan 1^\circ \tan 2^\circ} + 1} = \frac{-\tan 3^\circ + 1}{\tan 3^\circ + 1} = \frac{\tan 45^\circ - \tan 3^\circ}{1 + \tan 45^\circ \tan 3^\circ} = \tan(45^\circ - 3^\circ) = \tan 42^\circ.
\end{aligned}$$

Therefore, $\theta = 90^\circ - 42^\circ = 48^\circ$.

Answer: 48°

17. Find the sum

$$\sum_{n=1}^{84} \frac{1}{\sqrt{2n + \sqrt{4n^2 - 1}}}.$$

Solution: We try to write

$$\sqrt{2n+\sqrt{4n^2-1}} = \sqrt{a}+\sqrt{b}.$$

After squaring both sides we get

$$2n+\sqrt{4n^2-1} = a+b+2\sqrt{ab}$$

and by taking the system $a+b=2n$ and $4ab=4n^2-1$ we obtain one solution $a=n+\dfrac{1}{2}$ and $b=n-\dfrac{1}{2}$. Therefore, the following identity is true

$$\frac{1}{\sqrt{2n+\sqrt{4n^2-1}}} = \frac{1}{\sqrt{n+\frac{1}{2}}+\sqrt{n-\frac{1}{2}}} = \sqrt{n+\frac{1}{2}}-\sqrt{n-\frac{1}{2}},$$

which implies that

$$S_m = \sum_{n=1}^{m} \frac{1}{\sqrt{2n+\sqrt{4n^2-1}}} = \left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)$$
$$+\cdots+\left(\sqrt{m+\frac{1}{2}}-\sqrt{m-\frac{1}{2}}\right) = \sqrt{m+\frac{1}{2}}-\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}(\sqrt{2m+1}-1).$$

From here we get that $S_{84} = \dfrac{1}{\sqrt{2}}(\sqrt{169}-1) = 6\sqrt{2}$.

Answer: $6\sqrt{2}$ or $\dfrac{12}{\sqrt{2}}$

18. Evaluate the limit

$$L = \lim_{n\to\infty}(\lim_{x\to 0}(1+\tan^2 x+\tan^2 2x+\cdots+\tan^2 nx)^{\frac{1}{n^3x^2}}).$$

Solution: Let $f(x)=\tan^2 x+\tan^2 2x+\cdots+\tan^2 nx$. Then $\lim\limits_{x\to 0} f(x)=0$ and $\lim\limits_{x\to 0}(1+f(x))^{\frac{1}{f(x)}}=e$. Next we see that

$$\lim_{x\to 0}\frac{f(x)}{x^2} = \lim_{x\to 0}\left[\left(\frac{\tan x}{x}\right)^2+2^2\left(\frac{\tan 2x}{2x}\right)^2+\cdots+n^2\left(\frac{\tan nx}{nx}\right)^2\right]$$
$$=1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Therefore,

$$L = \lim_{n\to\infty}\left\{\lim_{x\to 0}\left[(1+f(x))^{\frac{1}{f(x)}}\right]^{\frac{f(x)}{x^2}}\right\}^{\frac{1}{n^3}} = \lim_{n\to\infty} e^{\frac{n(n+1)(2n+1)}{6n^3}} = e^{\frac{1}{3}}.$$

Answer: $e^{\frac{1}{3}}$ or $\sqrt[3]{e}$

19. Consider the sequence with general term

$$a_n = \sum_{k=1}^{n}\binom{n}{k}^2.$$

Find $\lim\limits_{n\to\infty}\sqrt[n]{1+a_n}$.

Solution: By representing the coefficient of x^n in two ways via the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$ we get

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

which implies that $1 + a_n = \binom{2n}{n}$. Next we use the Cauchy-D'Alembert Criterion to obtain

$$\lim_{n\to\infty} \sqrt[n]{1+a_n} = \lim_{n\to\infty} \sqrt[n]{\binom{2n}{n}} = \lim_{n\to\infty} \frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = 4.$$

Answer: 4

20. Evaluate the integral

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x \ln(1 + 2^{\sin x} + 3^{\sin x} + 6^{\sin x})dx.$$

Solution: If f is a continuous and even function on $[-1, 1]$ and $a > 0$ then

$$\int_{-1}^{1} \frac{a^x f(x)}{1+a^x}dx = \int_{-1}^{0} \frac{a^x f(x)}{1+a^x}dx + \int_{0}^{1} \frac{a^x f(x)}{1+a^x}dx = \int_{0}^{1} \frac{a^{-x} f(x)}{1+a^{-x}}dx + \int_{0}^{1} \frac{a^x f(x)}{1+a^x}dx = \int_{0}^{1} f(x)dx$$

(the latter equality follows from the identity $\frac{a^{-x}}{1+a^{-x}} = \frac{1}{1+a^x}$). Using this and integration by parts we get

$$\begin{aligned}\int_{-1}^{1} x\ln(1+a^x)dx &= \frac{x^2}{2}\ln(1+a^x)\Big|_{-1}^{1} - \frac{\ln a}{2}\int_{-1}^{1} \frac{x^2 a^x}{1+a^x}dx \\ &= \frac{1}{2}\ln(1+a) - \frac{1}{2}\ln\left(1+\frac{1}{a}\right) - \frac{\ln a}{2}\int_{0}^{1} x^2 dx = \frac{1}{2}\ln a - \frac{1}{6}\ln a = \frac{1}{3}\ln a.\end{aligned}$$

Therefore,

$$\begin{aligned}I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin x \ln((1+2^{\sin x})(1+3^{\sin x}))\cos x dx \\ &= \int_{-1}^{1} 2u[\ln(1+2^u) + \ln(1+3^u)]du = 2\int_{-1}^{1} u\ln(1+2^u)du + 2\int_{-1}^{1} u\ln(1+3^u)du = \frac{2}{3}\ln 6.\end{aligned}$$

Answer: $\frac{2}{3}\ln 6$