NUMERICAL ANALYSIS QUALIFIER

January 2003

Do all of the following five problems.

Problem 1. Let A and B be matrices in $\mathbb{R}^{n\times n}$, A be non-singular, and satisfy the inequality $||A^{-1}||_2||B||_2 \le q$ with a constant $q < 1$. Here $||\cdot||_2$ is the matrix norm subordinate to the Euclidean norm in $\mathbb{R}^n$.

(a) Show that $C = A + B$ is non-singular.
(b) Show that the iteration process $Ax^{j+1} = b - Bx^j$, $j = 0, 1, \dots$ converges for any x^0 to the solution of the system $Cx = b$. Give an estimate for the Euclidean norm of error $x^j - x$ in terms of q.
(c) Let $A = 2I$, where I is the identity matrix in $\mathbb{R}^{n\times n}$, and let B be the matrix with entries of -1 on the two main co-diagonals and zeros elsewhere. Estimate q in terms of n.

Problem 2. Consider the Cauchy problem $y' = f(t, y)$, $y(t_0) = y_0$ and the multistep formula

$$\eta_{n+1} = (1-a)\eta_n + a\eta_{n-1} + \frac{h}{12}\{(5-a)f_{n+1} + 8(1+a)f_n + (5a-1)f_{n-1}\}.$$

Here a is a real parameter, $f_n = f(t_n, \eta_n)$, $t_n = t_0 + nh$, and η_n is an approximation of $y(t_n)$.

(a) Show that the above method has a local truncation error which is $O(h^3)$ for all a.
(b) Find a value for a so that the local truncation error is $O(h^4)$ and check whether the resulting scheme satisfies the root condition.
(c) Define the region of absolute stability of a numerical method for solving the Cauchy problem for an ordinary differential equation. Define what it means for a multistep method to be A–stable. Is the method of Part (b) A–stable?

Problem 3. Consider the following two point boundary value problem:

$$-u''(x) + p(x)u'(x) + q(x)u(x) = f(x),\ 0 \le x \le 1,\ u(0) = u(1) = 0.$$

Define $x_j = jh$ for $j = 0, \dots, n$ and set $h = 1/n$. Let L_h be the $(n-1)\times(n-1)$ matrix corresponding to the finite difference scheme:

$$\begin{aligned} y_0 &= 0 \\ (L_h y)_j \equiv \frac{-y_{j-1} + 2y_j - y_{j+1}}{h^2} + p_j\frac{y_{j+1} - y_{j-1}}{2h} + q_j y_j &= f_j,\ j = 1, \dots, n-1, \\ y_n &= 0. \end{aligned}$$

Here $p_j = p(x_j)$, $q_j = q(x_j)$, $f_j = f(x_j)$, and y_j represents an approximation to $u(x_j)$.

(a) Assume that $q(x) \ge q_0 > 0$ for $x \in (0, 1)$ and that the step-size h satisfies

$$1 - h|p_j|/2 \ge 0 \ \text{ for } \ j = 1, \dots, n-1.$$

Show that L_h is non-singular and satisfies

$$\|L_h^{-1}\|_\infty \le q_0^{-1}$$

where $\|\cdot\|_\infty$ denotes the matrix norm subordinate to the L^∞ norm on $\mathbb{R}^{n-1}$.
(b) If τ_j, $j = 1, \dots, n-1$ is the local truncation error, then use the results of Part (a) to show that

$$|u(x_j) - y_j| \le q_0^{-1}\|\tau\|_\infty,\ j = 1, \dots, n-1.$$

Problem 4. Consider the space of linear functions on a triangle with three degrees of freedom given by the function values at the centers of the edges.

(a) Show that these nodal values form a unisolvent set.
(b) Suppose Ω is a polygonal domain in $\mathbb{R}^2$ and let $\mathbf{T} = \cup\tau$ be a triangulation of Ω. Let S_h be the approximation space associated with the degrees of freedom of Part (a). Prove or disprove: S_h is a subspace of $H^1(\Omega)$.
(c) Let u be the solution of the problem

$$u - \Delta u = f \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega.$$

Define

$$a_h(v, w) = (v, w) + \sum_{\tau \in \mathbf{T}} \int_\tau \nabla v \cdot \nabla w \, dx \quad \text{where} \quad (v, w) = \int_\Omega vw \, dx.$$

Let u_h be the unique function in S_h satisfying

$$a_h(u_h, w) = (f, w) \quad \text{for all} \quad w \in S_h.$$

Derive a representation for the error

$$\rho(\chi) \equiv (f, \chi) - a_h(u, \chi), \; \chi \in S_h,$$

in terms of integrals of the normal derivative of the solution u (and suitable expressions involving χ) along the edges of the triangulation.
(d) State an error estimate in the norm

$$\|u - u_h\|_h \equiv a_h(u - u_h, u - u_h)^{1/2}$$

in terms of the quantity $\rho(\chi)$ (you need not provide a bound for $\rho(\chi)$).

Problem 5. Consider the one dimensional wave equation:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \quad \text{for} \quad x \in (0, 1), t > 0, \\ u(0, t) = u(1, t) &= 0, \quad \text{for} \quad t > 0, \\ u(x, 0) = f(x), \; u_t(x, 0) &= g(x), \quad \text{for} \quad x \in [0, 1]. \end{aligned}$$

(a) Define the semi-discrete approximation to this problem based on a conforming approximation space $S_h^0 \subset H_0^1(0, 1)$.
(b) For $k > 0$, define the fully discrete scheme based on the time difference

$$\frac{\partial^2 u}{\partial t^2}(x, t_n) \approx \frac{u(x, t_{n+1}) - 2u(x, t_n) + u(x, t_{n-1})}{k^2}$$

where the term of the spatial derivative is evaluated at $t_n = nk$. This results in an approximate solution denoted U^n, ($U^n \in S_h^0$ and $U^n \approx u(\cdot, t_n)$, $n = 0, 1, \ldots$). Make sure that you define U^0 and U^1 appropriately.
(c) Derive a Courant condition which guarantees stability of the time stepping method of Part (b) above. You may assume that the spatial mesh is equally spaced so that the "mass" and "stiffness" matrices share the same eigenvectors.