Numerical Analysis Qualifier
Fall 1994

1. Attempt any five of the following problems. Each problem is 20 points. The points in the subproblems indicate the relative weight of that part.
2. Start each problem on a new sheet of paper and write on one side of each sheet of paper.

Problem 1. Let $||x||_1 = \sum_{i=1}^n |x_i|$ for $x \in R^n$. For any martix $A = \{a_{ij}\}_{ij=1}^n \in R^{n\times n}$ define the norm of A by

$$||A||_1 = \max_{x\neq 0} \frac{||Ax||_1}{||x||_1}.$$

Prove that

$$||A||_1 = \max_j \sum_{i=1}^n |a_{ij}| \quad \text{(column sum norm)}.$$

Problem 2.

1. (13 pts) Show that if the matrix A has real and positive eigenvalues then for any $\tau > 0$ the iteration method

$$x^{(m+1)} = x^{(m)} - \tau(A\frac{x^{(m+1)} + x^{(m)}}{2} - b)$$

   is convergent to the solution of $Ax = b$;
2. (7 pts) Determine the optimal parameter τ for which the convergence is maximal.

Problem 3.

1. (12 pts) Find the coeficients $a_0, \;\; a_1, \;\; a_2$ and a_3 and the nodes x_1 and x_2 in such a way that the quadrature

$$\int_{-1}^1 f(x)dx \approx a_0 f(-1) + a_1 f(x_1) + a_2 f(x_2) + a_3 f(1)$$

   has the highest possible algebraic degree of accuracy;
2. (3 pts) Find the algebraic degree of accuracy of the construced formula;
3. (5 pts) Consider the set of algebraic polynomials that are orthogonal on the interval $(-1, 1)$ with a wieght $1 - x^2$. Verify that the nodes of the obtained quadrature are zeros of the polynomial of second degree from that set.

Problem 4 . Consider the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad t > t_0, \quad y(t_0) = y_0,$$

and its approximation by the Runge-Kutta method

$$u_{n+1} = u_n + h(k_1 + 3k_2)/4, \quad u_0 = y_0, \quad t_{n+1} = t_n + h,$$

where $k_1 = f(t_n, u_n)$, $k_2 = f(t_n + \frac{2}{3}h, u_n + \frac{2}{3}hk_1)$, and u_n approximates $y(t_n) = y_n$.

1. (6 pts) Use Taylor's theorem to obtain an expression for the local truncation error of the method;

2. (10 pts) For $f(t, y) = t - y$, $t_0 = 0$, $y_0 = 1$ perform computations with step-size $h = 0.4$ and with step-size $h = 0.2$ to obtain two different approximations for $y(0.4)$. Apply the estimate for the local truncation error and the asymptotic expansion for the error to determine whether the obtained solution is 0.005-accurate, i.e. $|y(0.4) - u_1| \leq 0.005$;

3. (4 pts) If the required accuracy is not satisfied for the chosen step-size $h = 0.4$ explain how should you change the step-size in order to improve the accuracy.

Problem 5. The boundary value problem

$$-u'' + u = 0, \quad x \in (0, 1), \qquad -u'(0) + u(0) = 1, \quad u'(1) = 5$$

1. (4 pts) Derive the variational formulation of this problem and describe the energy space;

2. (12 pts) Using liner finite elements derive the Ritz system;

3. (4 pts) Estimate the local truncation error of the equation in the Ritz system that approximates the boundary condition at $x = 0$.

Problem 6. Consider the following problem for the transient heat equation:

$$u_t = u_{xx} + 10u_x + F(x, t), \quad for \quad x \in (0, 1), \quad t > 0,$$
$$u(0, t) = 0, \quad u(1, t) = 0, \quad for \quad t > 0,$$
$$u(x, 0) = f(x), \quad for \quad x \in (0, 1).$$

1. (5 pts) Write a finite difference approximation to this problem that is explicit in time and second order in space and estimate the local truncation error;

2. (11 pts) Using maximum principle type of argument find the exact Courant condition for stability and derive an a priori estimate for the solution in maximum norm;

3. (4 pts) Find an estimate for the error of the method.

Problem 7. Given the boundary value problem for $u(x, y)$:

$$-\Delta u \equiv -u_{xx} - u_{yy} = 0, \quad in \quad \Omega = (0, 2)^2,$$

$$u(x, 0) = x \text{ for } \; 0 \le x \le 2, \quad u(0, y) = y \text{ for } \; 0 \le y \le 2,$$

$$\frac{\partial u}{\partial n} = 1 \text{ on the remaining part of the boundary,}$$

where n is the outer normal to the boundary of Ω.

1. (5 pts) Give the weak formulation in H^1 to this problem. Let S^h be a finite element approximation of the Sobolev space H^1. Write down the Ritz-Galerkin system for the approximate solution to this problem in S^h. For S^h being the set of piece-wise polynomial functions of degree k state the error estimates for the finite element solution in H^1- and L^2-norms.

2. (13 pts) The square Ω is divided into 8 equal triangles and S^h is the set of the linear triangular finite elements based on this triangulation. Find the system of linear algebraic equations for the approximate values of u using S^h.

3. (2 pts) Verify that the exact solution of the boundary value problem $u(x, y) = x + y$ is also a solution to the algebraic problem.