Bounding the Number of Components of Polynomial Hypersurfaces

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That is, we are interested in the zero set of $\frac{1}{2}at^2 - d$.

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- In these cases, it helps to know how many solutions there are.

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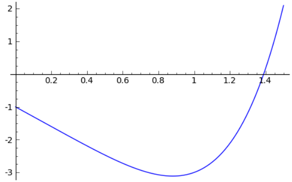

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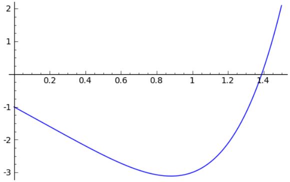

Idea: we can tell when to stop looking if we know how many roots there are.

Notation

Given $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and* $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$*, we define*

$$\mathbf{x}^a = x_1^{a_1} \cdots x_n^{a_n}$$

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as

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where $a_1 = (2, 0)$, $a_2 = (1, 1)$, and $a_3 = (0, 2)$.

For our purposes, we'll use the following definition of a polynomial:

Definition

An n-**variate** m-**nomial** is a polynomial in n variables with m terms, that is, a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f = \sum_{i=1}^{m} c_i \mathbf{x}^{a_i}$$

where $c_i \in \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $a_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{Z}^n$.

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Example

$f = 5x_1x_2^2 + 7x_2 + 3x_1^4 - 8x_1^3x_2 - x_2^5$ is a 2-variate 5-nomial.

And now for the objects of our interest:

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The **positive real zero set** of a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is the set $Z_+(f) = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0 \text{ and } f(\mathbf{x}) = 0\}$.

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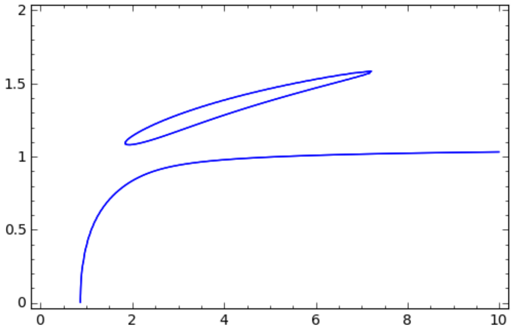
For polynomials in one variable, these are finite (unless the polynomial itself is 0). For multivariate polynomials, though, this need not be true.

Example

$g = x^5 - \frac{23}{20}x^6 + x^6y^2 - \frac{23}{20}x^4y^{10} + y^{25}$ has zero set



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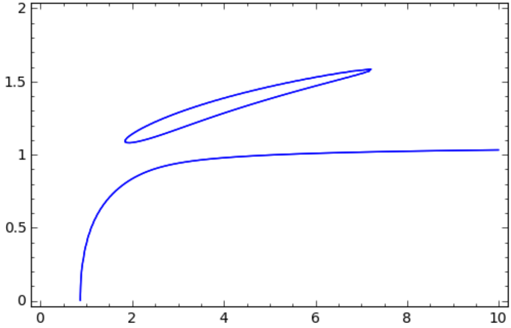

The **connected components** are the distinct curves in this set.
Compact components are closed and bounded.

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Consider $f = \frac{21}{20} - x^2y + x^3y^2 - x^4y^4 + \frac{3}{4000}x^5$.
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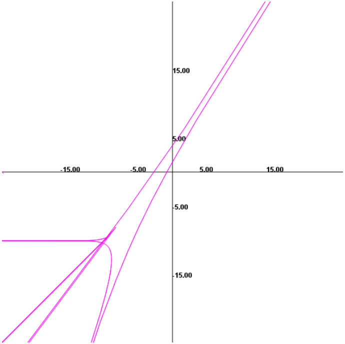

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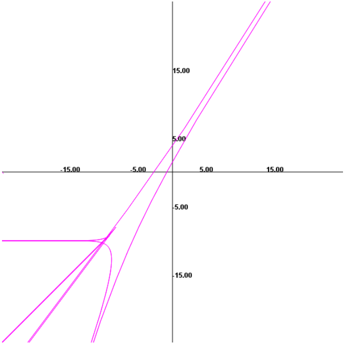

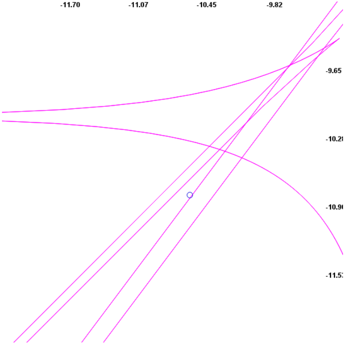

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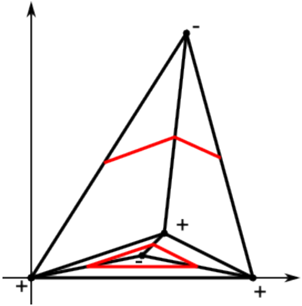

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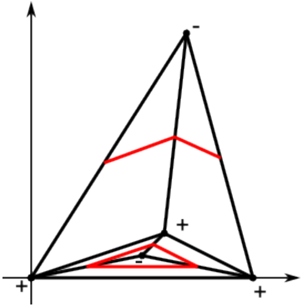

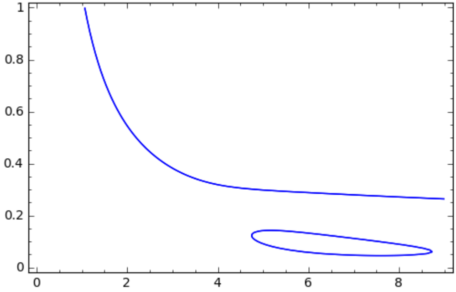

And the zero set is as above.

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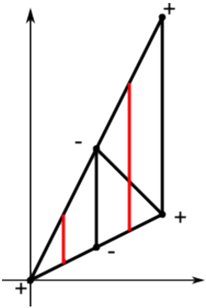
Example

Take $g = x^4y^2 - x^2y^4 - 3x^2y - 9xy^2 + 22$. Its viro diagram is



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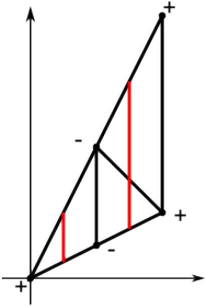

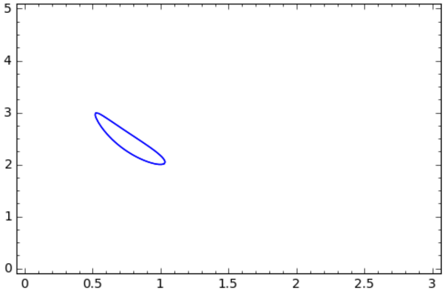

And here the zero set doesn't match.

Back to bounds

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Perrucci [3] found a way to bound compact components of 2-variate 4-nomials.

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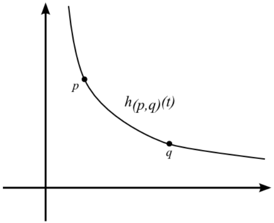

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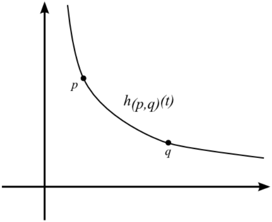

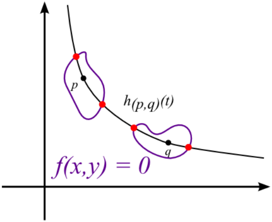

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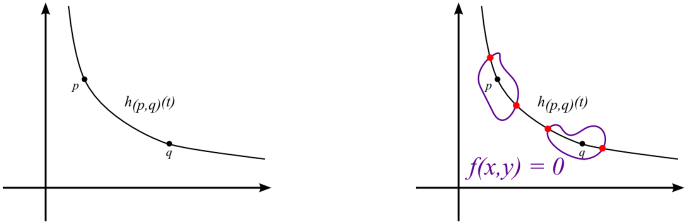

Using this method, we are working to improve the bound on 2-variate 5-nomials to less than 5.

Acknowledgments

Thanks to Dr. Rojas for guidance, background, and introducing this project.
Thanks to Korben Rusek for help; thanks to Daniel Perrucci, Frédérick Bihan and Frank Sottile, whose papers gave ideas for approaching this situation.
Thanks to Texas A&M University for hosting this REU program.

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