λ-Permutations of Conditionally Divergent Series II

Progress on Velleman's Problem

Speaker: Mamikon Gulian, Larson REU

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Definitions

$\sigma : N \to N$ is a $\lambda -$ *permutation* if

- $\forall \sum b_i$ convergent, $\sum b_{\sigma(i)}$ is convergent
- $\exists \sum a_i$ divergent, with $\sum a_{\sigma(i)}$ convergent.

$\iff$ in

- $\sigma([1, n]) = [c_1^n, d_1^n] \cup [c_2^n, d_2^n] \cup ... \cup [c_{b_n}^n, d_{b_n}^n]$
  $b_n < C$ (bounded block number)

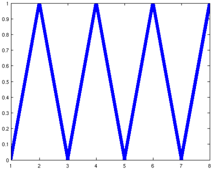
The set of all λ-permutations is denoted by Λ.

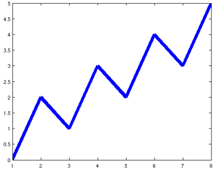
Velleman's Problem

Velleman (2006): Fix a conditionally divergent $\sum a_i$. Put

$$S = \{x \in R \mid \exists \sigma \in \Lambda \mid \sum a_{\sigma(i)} = x\}$$

Examples exist where $S = \mathbb{R}$ and $S = \emptyset$





Problem: is S *always* $\mathbb{R}$ or $\emptyset$?

- A finite subsequence B of consecutive terms of $\{a_i\}$ is a *block*. If all terms of B have the same sign, B is a *pure (positive or negative) block)*
  For B not pure, B is a *generalized* block.
- If $B = \{a_i, a_{i+1}, ..., a_{i+p}\}$ we define the *block sum* $|B|$ of B by

$$|B| = a_i + a_{i+1} + ... + a_{i+p}$$

Theorem A

Suppose $\sum a_i$ has ONE of the following properties

- There exists a sequence of disjoint blocks P_i of positive terms ("pure positive blocks") such that $|P_i| > A > 0$
- There exists a sequence of disjoint blocks N_i of negative terms ("pure negative blocks") such that $|N_i| < B < 0$

($|P_i|$ denotes sum of terms in P_i)

Then if S is not empty, $S = \mathbb{R}$.

Proof by Shifting Argument

If S is not empty, say $r \in \mathbb{R}$, then $\sum a_{\sigma(i)} = r$ for $\sigma \in \Lambda$.

- To increase the limit, shift blocks to the beginning of series and subsitute with other blocks.
- To decrease the limit, skip blocks at the beginning and substitute them for other blocks.

Drawbacks

- Existence of pure blocks are insufficient for classification ($\textit{when}$ is $S = \emptyset$, $\textit{when}$ is $S = \mathbb{R}$)
- Example: $\sum a_i$ conditionally divergent for which $S = \mathbb{R}$ due to existence of pure blocks (say $|P_i| = 1$).
  Define new series $\sum b_i$ by inserting $-\frac{1}{2^k}$ $(k = 1, 2, \ldots)$ between each term of the P_i.
  To each $\sigma \in \Lambda$ such that $\sum a_{\sigma(i)} = r$ there is a natural $\tau \in \Lambda$ such that $\sum a_{\tau(i)} = r - 1$.
  $\implies S = \mathbb{R}$ for $\sum b_i$.

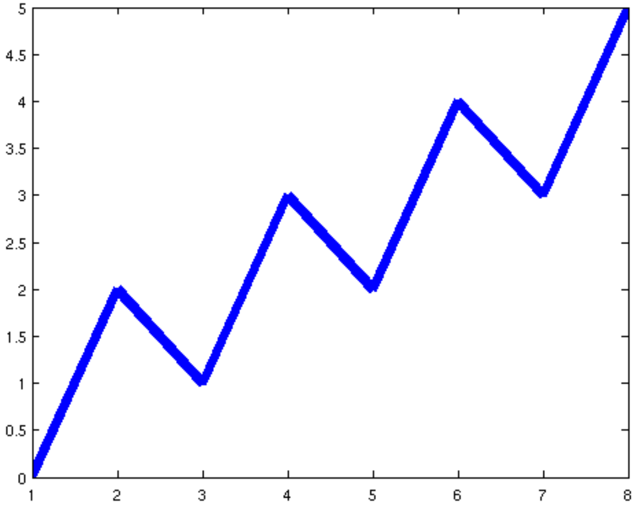
- A finite subsequence B of consecutive terms of $\{a_i\}$ is a *block*. (Sometimes *"generalized"*, *"impure" blocks*). The terms of B need not have the same sign.
- If $B = \{a_i, a_{i+1}, ..., a_{i+p}\}$ we define the *block sum* $|B|$ of B by

$$|B| = a_i + a_{i+1} + ... + a_{i+p}$$

- Let S_+ be the set of all $\delta \geq 0$ such that there are infinitely many *disjoint* blocks B_i such that $|B_i| \geq \delta$,
- Define $\alpha = \sup S_+$
- Let S_- be the set of all $\delta \leq 0$ such that there are infinitely many disjoint blocks B_i such that $|B_i| \leq \delta$.
- Define $\beta = \inf S_-$

"Unbalanced" Example

$\alpha = 2, \beta = -1, S = \emptyset$



Theorem B

$\alpha + \beta = 0$ or else S is empty.

Proof:

- Assume S is not empty: for some $\sigma \in \Lambda$, $\sum a_{\sigma(i)}$ converges.
- WLOG, take $\alpha + \beta > 0$. Take any δ such that, $\alpha + \beta > \delta > 0$
- For large enough n, the tail $\{a_i\}_{i=n}^{\infty}$ contains no blocks with sum $\leq -\alpha + \delta < \beta$. But it contains infinitely many blocks B_i with sum $\geq \alpha - \frac{\delta}{2}$.
- Hence the partial sums of the tail sequence successively get larger than $k\frac{\delta}{2}$, where k is the number of B_i passed, so $\sum a_i = +\infty$

Proof continued

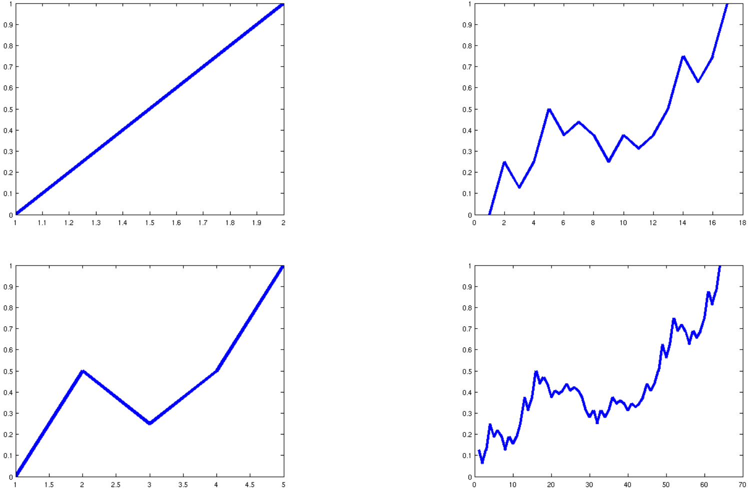
- Now write $\sigma([1, n]) = [p, q] \cup B_2 \cup \ldots \cup B_{b_n}$
- For large n, $p = 1$.
- As $n \to \infty$, $q \to \infty$.
- $\sum_{i=1}^{n} a_{\sigma(i)} = \sum_{i=1}^{q} a_i + \sum_{i=2}^{b_n} |B_i|$
- $\sum_{i=1}^{n} a_{\sigma(i)} \geq \sum_{i=1}^{q} a_i + C \inf |B|$
  (infimum is over all blocks occuring after d_1^n)
- Taking limits, $\lim \sum_{i=1}^{d_1^n} a_i = +\infty$ and $\liminf |B| = \beta$.
- $\sum_{i=1}^{\infty} a_\sigma(i) = +\infty$, a contradiction.

What cases are left?

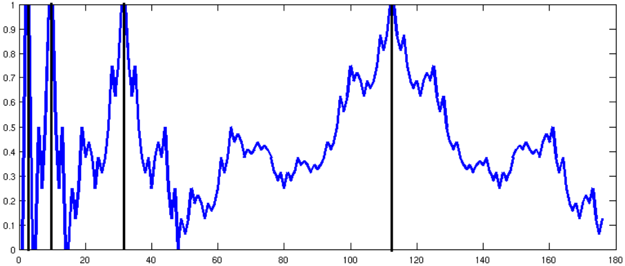
Three cases:

- $\alpha = 0 = \beta$ cannot occur, since $\sum a_i$ is not Cauchy.
- $\infty > \alpha = -\beta > 0$. (main effort)
  - Can we extend the shifting argument from the pure block case?
- $\infty = \alpha = -\beta$
  - Asyptotics, orders of growth of $+$,- blocks.
  - Too hard.

Challenges with bounded α, β: Fractal Oscillations



A Disappointment



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