Uncertainty and Information in Time-Frequency Analysis

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Preliminaries

$L^2(\mathbb{R})$ is the space of all functions from $\mathbb{R} \to \mathbb{C}$ such that their L^2 norm: $||f||_2 = (\int_{\mathbb{R}} |f(t)|^2 dt)^{1/2} < \infty$ is finite.

Definition

The Fourier Transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of a function f is defined as

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt.$$

Plancherel's Theorem: $||f||_2 = ||\hat{f}||_2$

Preliminaries continued

Discrete Setting:

Definition

Let $x \in \mathbb{R}^N$, i.e. $x = (x_t)_1^N = (x_1, x_2, \ldots, x_N)$. The Discrete Fourier Transform (DFT) of x is given by:

$$\mathcal{F}x(\omega) = \hat{x}(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} x_t \cdot e^{-2\pi i \omega t/N}, \omega = 1, 2, \ldots, N.$$

Plancherel's Theorem: $\sum_{t=1}^{N} |x_t|^2 = \sum_{\omega=1}^{N} |\hat{x}_\omega|^2$.

Classical Uncertainty Principle

Let $\Delta_f t = \left(\int_{\mathbb{R}} (t - t_0)^2 |f(t)|^2 dt\right)^{1/2}$ where $t_0 \in \mathbb{R}$.

Let $\Delta_f \omega = \left(\int_{\mathbb{R}} (\omega - \omega_0)^2 |\hat{f}(\omega)|^2 d\omega\right)^{1/2}$ where $\omega_0 \in \mathbb{R}$.

Theorem (Heisenberg's Inequality)

If $f \in L^2(\mathbb{R})$ with $||f||_2 = 1$, then

$$\Delta_f t \cdot \Delta_f \omega \geq \frac{1}{4\pi}$$

Quantum Mechanics: $\Delta_f t$ = position "uncertainty",
$\Delta_f \omega$ = momentum "uncertainty"

Uncertainty Principle of Donoho and Stark

Definition

A function f is ϵ-concentrated on a set T if

$$||f - \chi_T f||_2 \leq \epsilon$$

where χ_T is the characteristic function of the set T.

Theorem (Donoho/Stark 1989)

Suppose f is ϵ_T-concentrated on T, and its Fourier transform $\hat{f}$ is ϵ_W - concentrated on a set W with $||f||_2 = 1$. Then

$$m(T) \cdot m(W) \geq (1 - \epsilon_T - \epsilon_W)^2.$$

Information Theory

Introduced by C.E. Shannon's 1948 paper: "A Mathematical Theory of Communication"

Sentence 1: “The sun will set in the west tomorrow"

Sentence 2: “There will be a solar eclipse tomorrow"

Which has more information?

Caveat- Received message: “WZHSLNRU?@TG"

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Three Intuitive Postulates for Information:

1. If E, F are events such that $P(E) \leq P(F)$, then $I(E) \geq I(F)$.
2. If E, F are independent events, $I(E \cap F) = I(E) + I(F)$.
3. For all events E, $I(E) \geq 0$.

(Shannon 1948) The only function that satisfies 1,2,3 is of the form:

$$I(E) = -K log_a(P(E))$$

where a, K are positive constants.

Information as a Random Variable

Consider a discrete random variable $X : \mathcal{S} \to \{x_1, \ldots, x_n\} \subset \mathbb{R}$ with associated probability distribution specified by $p_i = P(X = x_i)$.

Example: Let $\mathcal{S} = \{\text{heads}, \text{tails}\}$. Then $X : \mathcal{S} \to \{0, 1\}$ is a random variable where $X(\text{heads}) = 1$ and $X(\text{tails}) = 0$ with associated probabilities $p_0 = p_1 = \frac{1}{2}$.

Warning: $X = 1$ is commonly written instead of $X(\cdot) = 1$.

Definition

The information of a random variable X is given by $I(X) : \{x_1, \ldots, x_n\} \to \mathbb{R}$ by $I(X) = -\log_2(P(X))$. The units of information with respect to $\log_2$ are called bits.

Entropy as a Measure of Uncertainty

Examples of Calculating Information of Events:

I(“coin landing heads")
$= -\log_2(1/2) = 1$ bit

I(“card being Ace of Spades")
= - $\log_2(\frac{1}{52}) \approx 5.70$bits

Definition (Shannon 1948)

The entropy of a random variable X is the expected value of $I(X)$ given by

$$H(X) = \mathbb{E}(I(X)) = -\sum_{j=1}^{n} p_j \log_2(p_j).$$

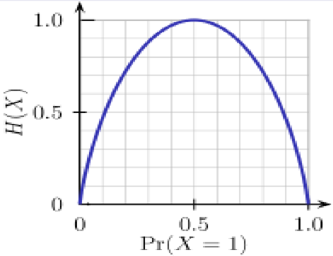


Figure: Entropy of a “Weighted" Coin Flip

Hirschman Uncertainty Principle

Let x_t, $\hat{x}_\omega \in \mathbb{R}^N$ such that $||x|| = 1$.

Let X, Y be random variables who map into $\{1, 2, \ldots, N\}$ with associated probability distributions given by $P(X = i) = |x_i|^2$ and $P(Y = i) = |\hat{x}_i|^2$.

Theorem (Hirschman's Uncertainty Principle (Dembo et al. 1991))

Let x_t and $\hat{x}_\omega$ be a Fourier transform pair such that $||x|| = 1$. Then defining random variables X, Y as given above, we have

$$H(X) + H(Y) \geq \log_2(N).$$

New Approach: Approximate Concentration of Entropy

Let x_t and $\hat{x}_\omega$ be a Fourier transform pair in $\mathbb{R}^N$ such that $||x|| = 1$ and X and Y be defined as before.

Let $T \subseteq \{1, \ldots, N\}$. Define $H(X|_T) = -\sum_{j \in T} p_j \log_2(p_j)$.

Definition

X is ϵ-concentrated in entropy to a set $T \subseteq \{1, 2, \ldots, N\}$ if

$$H(X) - H(X|_T) = -\sum_{j \notin T} p_j \log_2(p_j) \leq \epsilon.$$

Question: Are there lower bounds for $H(X|_T), H(Y|_W)$ that can be established?

Numerical Simulations

$H(X) + H(Y) =$ Sum of Entropies,
$H(X|_T) + H(Y|_W) =$ Sum of Approximate Entropies

Figure: $\epsilon_T = \epsilon_W = 1/10$

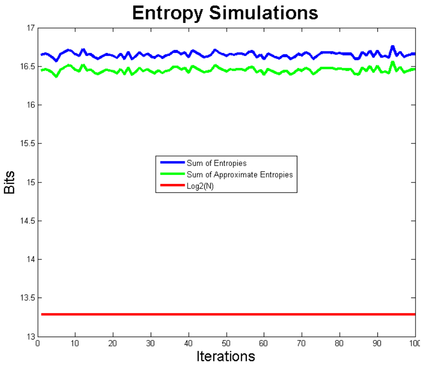
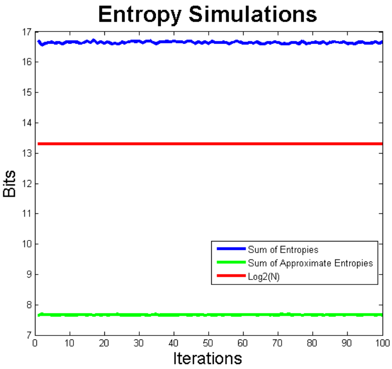


Figure: $\epsilon_T = \epsilon_W = 5$



An Uncertainty Result for Approximate Concentration of Entropy

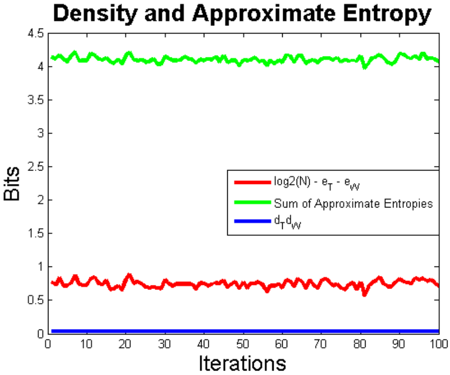
Theorem

Let x_t and $\hat{x}_\omega$ be a Fourier transform pair in $\mathbb{R}^N$ such that $||x|| = 1$ and two random variables X, Y who share the same range, where $P(X = i) = |x_i|^2$ and $P(Y = i) = |\hat{x}_i|^2$. Suppose X is ϵ_T-concentrated in entropy to a set T, and Y is ϵ_W-concentrated in entropy to a set W. Then we have

$$\log_2(N) - \epsilon_T - \epsilon_W \leq H(X|_T) + H(Y|_W).$$

Density of the sets T and W

We define the density of T to be $d_T = \frac{N_T}{N}$ where N_T is the number of non-zero entries in T. Similarly, we define the density of W, $d_W = \frac{N_W}{N}$.



Results and Conjectures

Let X and Y be defined for the unit-normalized Fourier transform pair $x, \hat{x}$ as given before.

Theorem

Let X be ϵ_T-concentrated in entropy to a set T, Y be ϵ_W-concentrated in entropy to a set W. Then for $N \geq 2^{1+\epsilon_T+\epsilon_W}$,

$$d_T d_W \leq \log_2(N) - \epsilon_T - \epsilon_W \leq H(X|_T) + H(Y|_W).$$

We also know that $1 - \epsilon_T - \epsilon_W \leq \log_2(N) - \epsilon_T - \epsilon_W$. The following conjecture is suggested by numerical simulations:

Conjecture

$$1 - \epsilon_T - \epsilon_W \leq d_T d_W.$$

References

1. A. Dembo, T.M. Cover, J.A. Thomas, *Information Theoretic Inequalities*, IEEE Transactions On Information Theory, Vol. 37, No.6, 1991.
2. D. L. Donoho and P. B. Stark. "Uncertainty Principles and Signal Recovery". SIAM Journal of Applied Mathematics 49 (1989), 906-931.
3. Gröchenig, Karlheinz."Foundations of Time-Frequency Analysis". Series: Applied and Numerical Harmonic Analysis, Birkhaüser, 2001

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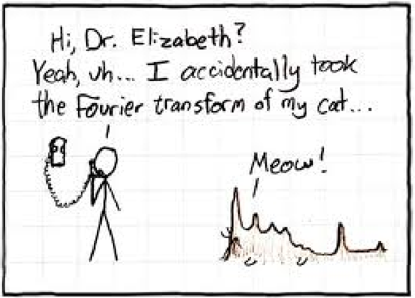

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Appendix

Theorem (Heisenberg's Inequality)

If $f \in L^2(\mathbb{R})$ with $||f||_2 = 1$, then

$$\left(\int_{\mathbb{R}} (t - t_0)^2 |f(t)|^2 dt\right)^{1/2} \cdot \left(\int_{\mathbb{R}} (\omega - \omega_0)^2 |\hat{f}(\omega)|^2 d\omega\right)^{1/2} \geq \frac{1}{4\pi}$$

Lemma

Let A, B be self-adjoint operators on a Hilbert space $\mathcal{H}$. We define the commutator of A, B to be $[A, B] := AB - BA$. Then we have that

$$||(A - a)f|| \cdot ||(B - b)f|| \geq \frac{1}{2} |\langle [A, B]f, f \rangle|$$

for $a, b \in \mathbb{R}$ and f in the domain of $AB \cap BA$.

Proof of Lemma

Proof.

$$
\begin{aligned}
|\langle [A,B]f, f\rangle| &= |\langle ((A-a)(B-b)-(B-b)(A-a))\, f, f\rangle| \\
&= |\langle (B-b)f, (A-a)f\rangle - \langle (A-a)f, (B-b)f\rangle| \\
&\leq |\langle (B-b)f, (A-a)f\rangle| + |\langle (A-a)f, (B-b)f\rangle| \\
&\leq 2||(B-b)f|| \cdot ||(A-a)f||
\end{aligned}
$$

from which the lemma follows. □

Proof of Heisenberg's Inequality

With this lemma, we may continue with the proof of the theorem. Let the operators $A, B \in B(L^2(\mathbb{R}))$ by

$$Af = tf(t), B = \frac{1}{2\pi i} f'(t).$$

A, B are self-adjoint operators. By the lemma, we have then that

$$||(A-a)f|| \cdot ||(B-b)f|| \geq \frac{1}{2}|\langle [A,B]f, f\rangle|.$$

Observe that

$$\frac{1}{2}|\langle [A,B]f, f\rangle| = \frac{1}{2}|\int_{\mathbb{R}} \frac{1}{2\pi i}|f(t)|^2 dt| = 1/4\pi.$$

Then,

$||(B-b)f|| = ||\mathcal{F}(B-b)(f)|| = \left(\int_{\mathbb{R}}(\omega-\omega_0)^2|\hat{f}(\omega)|^2 d\omega\right)^{1/2}$ and $||(A-a)f|| = \left(\int_{\mathbb{R}}(t-t_0)^2|f(t)|^2 dt\right)^{1/2}$. QED.