Texas A&M REU 2017
Number Theory Group

Zeros of Eisenstein Series Arising from Dirichlet Characters

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Acknowledgements

- Dr. Young
- Victoria Jakicic
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Special Linear Group

The **special linear group** of degree 2 with coefficients in $\mathbb{Z}$, denoted $\mathrm{SL}_2(\mathbb{Z})$ is defined as

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

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Let $\mathcal{H}$ denote the upper half-plane

$$\mathcal{H} = \{x + iy \in \mathbb{C} : y > 0\}.$$

Fundamental Domain

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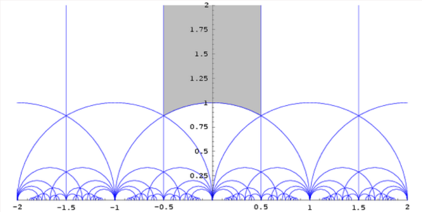
The **fundamental domain** $\mathcal{F} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ is shown as



Modular Forms

A map $f : \mathcal{H} \to \mathbb{C}$ is called a **modular form** of weight k if
1) f is holomorphic on $\mathcal{H}$
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Weakly modular of weight k means that

$$f(\gamma(z)) = (cz+d)^k f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Modular Forms on Congruence Subgroups

We define the **slash operator of weight** k to be

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The space of these modular forms is denoted $\mathcal{M}_k(\Gamma_0(N))$.

An Example

The classical **Eisenstein series** is given as

$$E_k = \frac{1}{2} \sum_{\gcd(c,d)=1} \frac{1}{(cz+d)^k}$$

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Our Problem (Part 1)

Where do Eisenstein series on $\Gamma_0(N)$ vanish?

Cusp Forms and Equidistribution

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Our Problem (Part 2)

What structure do our zeros display?

Recall From Last Time...

A **Dirichlet character** modulo *n* is a map $\chi : \mathbb{Z} \to \mathbb{C}$ which is
· totally multiplicative, that is, $\chi(1) = 1$ and $\chi(mn) = \chi(m)\chi(n)$ for all integers *m*, *n*
· periodic modulo *n*
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Given a primitive Dirichlet character χ_1 modulo q_1, and a primitive Dirichlet character χ_2 modulo q_2, we have the associated Eisenstein series of weight k:

$$E_{\chi_1,\chi_2,k}(z) = \sum_{(c,d)=1} \frac{\chi_1(c)\chi_2(d)}{(cq_2 z + d)^k} \in \mathcal{M}_k(\Gamma_0(q_1 q_2))$$

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Within a certain horizontal strip, $E_{\chi_1,\chi_2,k}(z)$ is dominated by just a few terms.

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We assume q_2 is prime, and let a be an integer such that a and $a + 1$ are coprime to q_2.

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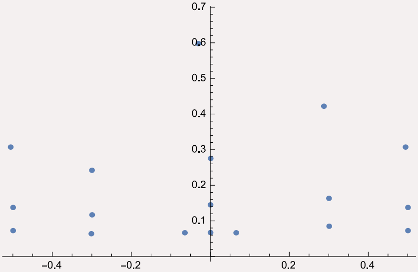
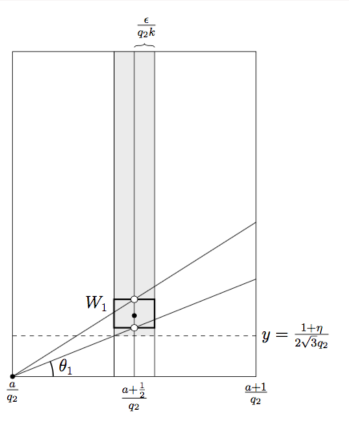


Figure: For $k = 10$, $q_1 = 3$, $q_2 = 5$

Zooming In


$\frac{\epsilon}{q_2 k}$
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$y = \frac{1+\eta}{2\sqrt{3}q_2}$
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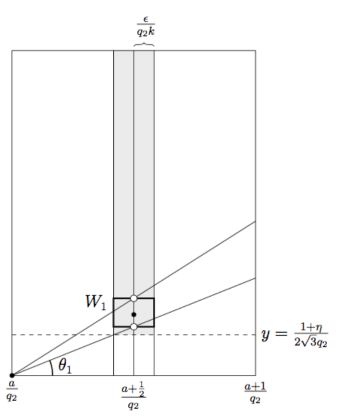

Letting θ denote the angle of z from the point $\frac{a}{q_2}$, we have proved that the zeros become distributed evenly with respect to θ as $k \to \infty$.

Distribution With Respect to θ

Let $z = x + iy$, so in a small strip around $\frac{a+1/2}{q_2} - \frac{\epsilon}{q_2 k} \leq x \leq \frac{a+1/2}{q_2} + \frac{\epsilon}{q_2 k}$, we have that the main terms are

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If θ satisfies the above equation, so does $\theta + \frac{n\pi}{k}$ for any $n \in \mathbb{N}$.

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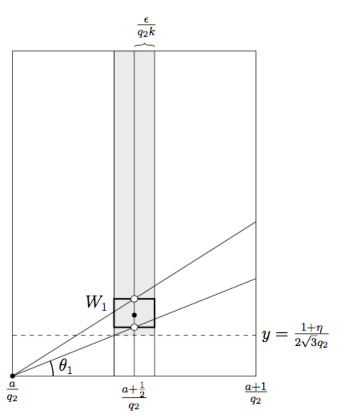
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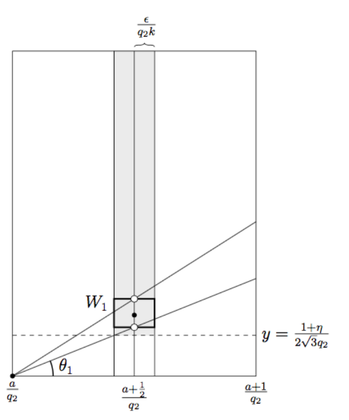

We look in small regions W_n around each zero of $g_a(z)$.

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In each W_n where $1 \leq n \leq m - 1$, both $g_a(z)$ and $E_{\chi_1,\chi_2,k}(z)$ have one zero when k is sufficiently large.

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Proposition

When $q_1 > 3$, all these zeros are $\Gamma_0(q_1 q_2)$-inequivalent. That is, there does not exist a $\gamma \in \Gamma_0(q_1 q_2)$ that maps one zero to another.

Theorem

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For k sufficiently large, $E_{\chi_1,\chi_2,k}(z)$ has $m = \frac{k}{3} + O\left(\sqrt{k}\log^2(k)\right)$ zeros tending to the vertical line $\Re(z) = \frac{a+1/2}{q_2}$ which are $\Gamma_0(q_1 q_2)$-inequivalent and become distributed with respect to their angle from the point $\frac{a}{q_2}$.

The Structure of Zeros

The zeros we have found vary by height on the order of $O\left(\frac{1}{k}\right)$, and there are $\varphi(q_2)$ lines of them in $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

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Conjecture

As q_2 tends to infinity, and k tends to infinity much slower, the zeros of $E_{\chi_1,\chi_2,k}(z)$ equidistribute when they are mapped back to the fundamental domain $\mathcal{F}$.

Thank you!

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