ON CLASSIFICATION OF (WEAKLY INTEGRAL) MODULAR CATEGORIES BY DIMENSION

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BACKGROUND

The question

We are looking at Categories with Frobenius-Perron dimension $4q^2$, $4p^2q$ and 2^n where $n = 5, 6$ and p, q are primes

Why do we care about classification?

Classifying fusion and modular categories has importance in

- physics including quantum computing
- topological quantum field theory
- conformal field theory,
- subfactor theory,
- representation theory of quantum groups and others

Previous Results

Theorem (BRUILLARD,PLAVNIK, et. al.)

There is classification of modular categories of dimensions pq^4, *when* p^2q^2 *is odd,* 2^3 *and* 2^4 *[1]*

Some Definitions

Definition

*A **modular category** is a non-degenerate pre-modular braided fusion category*

What is a fusion category?

A **category** consists of objects, arrows (morphisms) between the objects and a composition map ($Hom(y, z) \times Hom(x, y) \rightarrow Hom(x, z)$) with

- Associativity
- An identity homomorphism

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4. *$\mathscr{C}$ is semi-simple ($x = \oplus m_i x_i$ where x_i simple)*
5. *$\mathscr{C}$ is "finite"*
6. *$\mathbb{1}$ is simple*

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- $Hom_C(y, x)$ *is a* $\mathbb{C}$*-Vector space*
- *There exists direct sums*

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- *a: the family of natural isomorphism of associativity regarding tensor product*
- $\mathbb{1}$*: : an identity element is in Obj(*$\mathscr{C}$*)*
- *And for all x in* $\mathscr{C}$*:* ℓ *and r are the family of natural isomorphisms such that*
  - $\ell_x : \mathbb{1} \otimes x \widetilde{\to} x$
  - $r_x : x \otimes \mathbb{1} \widetilde{\to} x$

Definition of fusion categories

Definition

*A category is **rigid** if for every x there is a left and right dual.*

Definition

*A category is **semi-simple** when all objects in the category can be written as a direct sum of simple objects.*

Definition

*One thing that occurs when a category is **finite** is that there are a finitely many simple objects (up to isomorphisms).*

Example of a fusion category

An example of a fusion category is $\mathrm{Rep}(G)$, the category of finite dimensional complex representations of a finite group G. The objects are the representations and the arrows are intertwining maps.

What is Dimension?

There are a finite number of simple objects X_i (up to isomorphism). They all have a Frobenius-Perron dimension

$$FPDim(\mathscr{C}) = \sum_{k=0}^{r-1}(FPDim(x_k)^2)$$

Some important properties include:

$$FPDim(x \otimes y) = FPDim(x) \cdot FPDim(y)$$

$$FPDim(x \oplus y) = FPDim(x) + FPDim(y)$$

$$FPDim(\mathbb{1}) = 1$$

$$(\text{FPDim}(X_i))^2 | \text{FPDim}(\mathscr{C})$$

De-equivariantization

Let $\mathscr{B}$ be the subcategory of $\mathscr{C}$ generated by a self-dual invertible g. If $Z_2(\mathscr{B}) = Rep(\mathbb{Z}_2)$ then we can **de-equivariantize** the category and get a fusion category C_G with $\text{FPDim}(C_G) = \text{FPDim}(\mathscr{C})/2$

If an object x is stabilized by g, then in C_G there are two objects with dimension $\text{FPDim}(x)/2$

If an object y is mapped to an object w, then in C_G there is one object of dimension $\text{FPDim}(y) = \text{FPDim}(w)$

CURRENT PROGRESS

$\mathrm{FPDim}(\mathscr{C}) = 4q^2$

Let $\mathscr{C}$ be a modular category of Frobenius-Perron dimension $4q^2$

Then $\mathrm{FPDim}(x_i) \in \{1,\ 2,\ q,\ 2q,\ \sqrt{2},\ q\sqrt{2},\ \sqrt{q},\ 2\sqrt{q},\ \sqrt{2q}\}$

We are able to find the possible break down of the category based on the number of invertible objects and can eliminate or classify them.

The option in this case are

- 2
- $2q$
- $2q^2$

$\mathrm{FPDim}(\mathscr{C}) = 4q^2$, a $= 2$

In the integral component there are 2 invertible objects and $\frac{q^2-1}{2}$ simple object of dimension 2.

For the non-integral component there are four possibilities

- q^2 simple object of dimension $\sqrt{2}$
- 1 simple object of dimension $q\sqrt{2}$
- j simple objects with dimension $2\sqrt{q}$ and $2(q-2j)$ with dimension $\sqrt{q}$ $\quad$ j is a positive integer less than $\frac{q}{2}$
- q simple objects with dimension $\sqrt{2q}$

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In the remaining case C_G has q^2 invertibles and q simple objects dimension $\sqrt{q}$

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Consider the object $\mathbb{1}$. Since $\mathbb{1} \otimes g = g$, meaning g does not stabilize it. There is 1 invertible object in C_G

$\mathrm{FPDim}(\mathscr{C}) = 4q^2$, a $= 2$

Let Y_i be a simple object of dimension 2. Since $Y_i \otimes Y_i^* = \mathbb{1} \oplus g \oplus Y_k$ we know that g must stabilize all the simple objects of dimension 2.

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By collecting all the simple objects in C_G we get q^2 invertibles and q simple objects dimension $\sqrt{q}$

$\mathrm{FPDim}(\mathscr{C}) = 4q^2$, $a = 2q$

In the integral component there are q components with 2 invertible objects and $\frac{q-1}{2}$ simple object dimension 2.

For the non-integral component there are three possibilities

- q components with q simple object of dimension $\sqrt{2}$
- q components with 2 objects with dimension $\sqrt{q}$
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In the integral component there are q^2 components with 2 invertibles

The only choice of non-integral component is q^2 components with 1 object of dimension $\sqrt{2}$

This is a Generalized Tambara-Yamagami Category, which is well studied.

The Final case

Recall the case where C_G has q^2 invertibles and q simple objects of dimension $\sqrt{q}$

Since the integral component is modular and pointed we can say that $\mathscr{C}$ is a Gauging of $(\mathscr{C}_{int})_{\mathbb{Z}_2}$

$\mathrm{FPDim}(\mathscr{C}) = 2^5$

By similar methods we can find that any category with $\mathrm{FPDim}(\mathscr{C}) = 2^5$ are as follows

$$\mathscr{C} = B \boxtimes \mathcal{I} \boxtimes \mathcal{I}$$

$$\mathscr{C} = \mathcal{I} \boxtimes D$$

FUTURE WORK

Future work

Look into other dimensions

- $4p^2q$
- $4p^2q^2$
- 2^n

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