Max Intersection-Complete Codes

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Motivation

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- Place cells represent an animal's location
- Multiple place cells can fire at once

Notation

Definition

A **neural code** $\mathcal{C}$ on n neurons is a set of subsets of $[n]$.

- Given n neurons, we build neural codes from their respective *receptive fields*, living in $\mathbb{R}^d$.
- The receptive field of a neuron i is denoted U_i.

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- The receptive field of a neuron i is denoted U_i.
- On 5 neurons, one codeword could be $\{2,4\}$; this is where the receptive fields U_2 and U_4 overlap; we write this as 24.

Neural Codes and Convexity

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- Certain types of codes are known to be convex, notably max intersection-complete codes.

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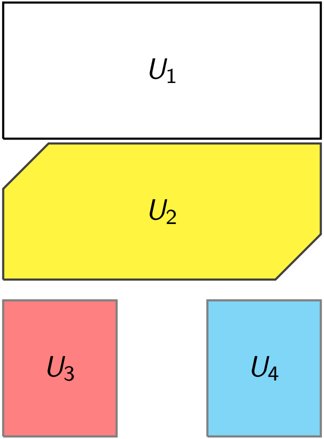
Definition

The **maximal code** on n neurons is $\mathcal{C}_{max}(n) = \{\sigma : \sigma \subseteq [n]\}$.

Max Intersection-Complete Codes



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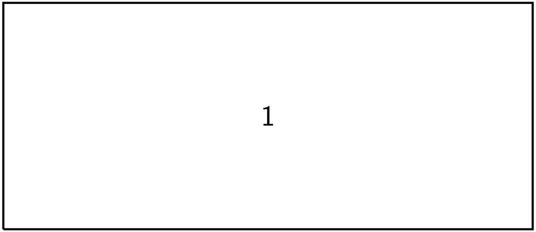

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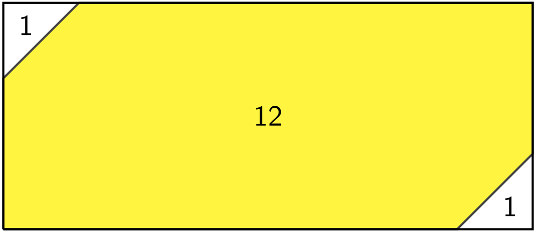

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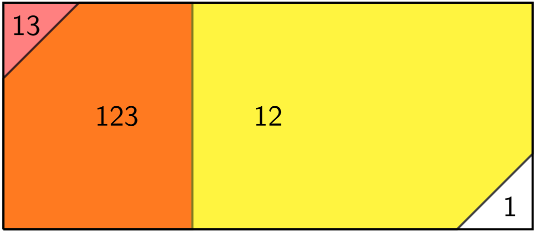

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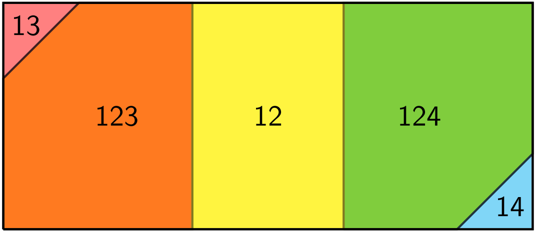

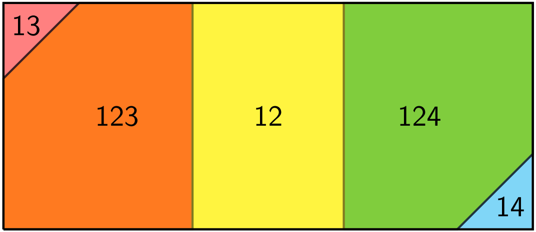

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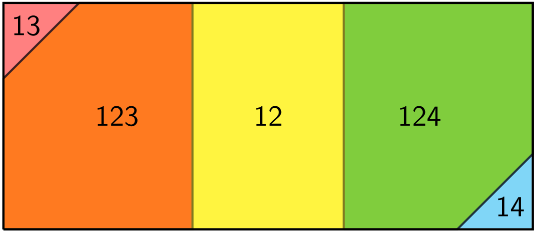

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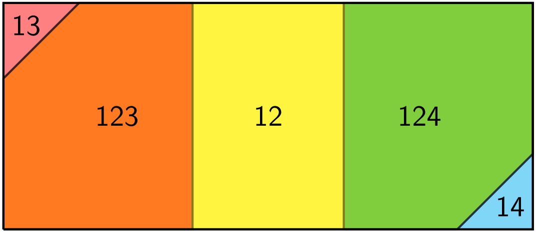

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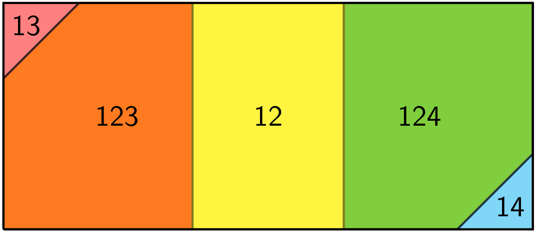

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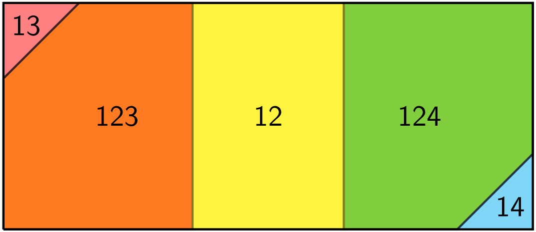

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Neural Ideals

From a neural code $\mathcal{C}$, we obtain its neural ideal $J_{\mathcal{C}}$, defined to be

$$J_{\mathcal{C}} := \langle \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 - x_j) : \sigma \notin \mathcal{C}, \tau = [n] - \sigma \rangle.$$

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In our example, 24 is not a codeword of $\mathcal{C}$, so

$$x_2 x_4 (1 + x_1)(1 + x_3) \in J_{\mathcal{C}}$$

The Canonical Form

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- Type 1 relations: $\prod_i x_i$
- Type 2 relations: $\prod_i x_i \prod_j (1 - x_j)$

- If a Type 1 relation $x_{a_1} \dots x_{a_n}$ is in the CF of $J_{\mathcal{C}}$, then the codeword $c = a_1 \dots a_n$ is not in $\mathcal{C}$, nor is any codeword containing c.
- If a Type 2 relation $x_{a_1} \dots x_{a_n}(1 - x_{b_1}) \dots (1 - x_{b_m})$ is in the CF, then

$$\bigcap_{i \in \{a_1, \dots, a_n\}} U_i \subseteq \bigcup_{j \in \{b_1, \dots, b_m\}} U_j$$

Canonical Form Example

Recall our code $\mathcal{C} = \{123, 124, 12, 14, 13, \emptyset\}$.

Here,
$CF(J_{\mathcal{C}}) = \{x_2(1-x_1), x_3(1-x_1), x_4(1-x_1), x_3x_4, x_1(1-x_2)(1-x_3)(1-x_4)\}$.

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Because $x_3x_4 \in CF(J_{\mathcal{C}})$, we can't have $34 \in \mathcal{C}$, nor can we have $134, 234$, or $1234 \in \mathcal{C}$.

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Further, an element like $x_2(1-x_1)$ tells us that $U_2 \subseteq U_1$.

Canonical Form Example

Similarly, because $x_1(1-x_2)(1-x_3)(1-x_4) \in CF(J_{\mathcal{C}})$, we have that $U_1 \subseteq U_2 \cup U_3 \cup U_4$.

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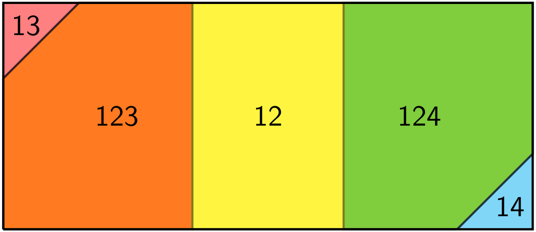

An Existing Signature for Intersection-Complete Codes

The following theorem gives a signature in the canonical form for intersection-complete codes:

Theorem (Curto, Gross, et al. 2015)

A code $\mathcal{C}$ is intersection-complete if and only if $CF(J_{\mathcal{C}})$ contains only monomials and pseudomonomials of the form $(1 - x_j)\prod_i x_i$.

Question

Research Question

Does there exist a signature in the canonical form for maximum intersection-complete codes?

Finding the Facets

We have been able to develop an algorithm for finding the facets of a code $\mathcal{C}$ from $CF(J_{\mathcal{C}})$.

We use the fact that if a monomial appears in $CF(J_{\mathcal{C}})$ then no codeword containing the indices of that monomial appears in $\mathcal{C}$.

Example of Facet Algorithm

Recall our earlier example: $\mathcal{C} = \{\mathbf{123}, \mathbf{124}, 12, 13, 14, \emptyset\}$.

The only monomial in $CF(J_{\mathcal{C}})$ is x_3x_4.

On 4 neurons,
$\mathcal{C}_{max} = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234\}$.

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Removing all codewords eliminated by this monomial gives us
$\mathcal{C}'_{max} = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, \sout{34}, 123, 124, \sout{134}, \sout{234}, \sout{1234}\}$.

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$\mathcal{C}'_{max} = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, \mathbf{123}, \mathbf{124}\}$.

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$\mathcal{C}'_{max} = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, \mathbf{123}, \mathbf{124}\}$.

We see that the facets of $\mathcal{C}$ and our reduced $\mathcal{C}'_{max}$ are the same.

Sufficient Condition for Non-maximality

Proposition (Franke-H)

Let $\mathcal{C}$ be a neural code, $J_{\mathcal{C}}$ be its neural ideal, and $CF(J_{\mathcal{C}})$ be the corresponding canonical form. If there exist $\tau \subset [n]$ and $\sigma \subseteq [n] - \tau$ such that $\prod_{i \in \tau} x_i \in CF(J_{\mathcal{C}})$ and $\prod_{j \in \sigma} x_j \prod_{i \in \tau} (1 - x_i) \in CF(J_{\mathcal{C}})$, then $\mathcal{C}$ is not convex.

Corollary

For a code to be max intersection-complete, it cannot have the above condition.

Example

Let $\mathcal{C} = \{4, 5, 1234, 1235, \emptyset\}$ be a code on five neurons. The canonical form contains both x_4x_5 and $x_1(1 - x_4)(1 - x_5)$.
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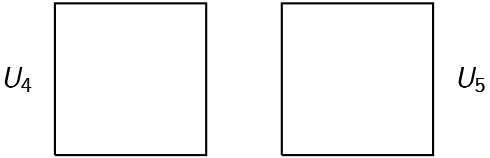

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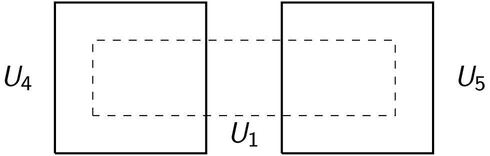

An Algorithm for Determining Missing Codewords

1. Pick a complex pseudomonomial

This is a pseudomonomial with multiple $(1 - x_j)$ factors, e.g., $x_i(1 - x_{j_1}) \dots (1 - x_{j_m})$. Write $\cap_{k \in [m]} ij_k = i$.

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Neurons which always fire together are equivalent, e.g. if $x_i(1 - x_j)$ and $x_j(1 - x_i) \in CF(J_{\mathcal{C}})$, then neurons i and j are equivalent.

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3. Add all other possible neurons not prevented by monomials.

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2. $x_4(1 - x_5), x_5(1 - x_4), x_2(1 - x_3), x_3(1 - x_2) \in CF(J_{\mathcal{C}})$, so

$U_4 = U_5$ and $U_2 = U_3$
Potentially, we then have $2345 \cap 236$

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3. x_4x_6, x_5x_6 are the only monomials in $CF(J_{\mathcal{C}})$, so we can add 1 to 2345 and 236 to get $1236 \cap 12345 = 123$

Acknowledgments

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