ALGEBRAIC SIGNATURES OF AN OBSTRUCTION TO CLOSED CONVEXITY AND SUNFLOWERS

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July 2020

Abstract

Neural codes dictate intersection relationships for the receptive fields they generate. According to experimentation, these sets should be convex. However, local and non-local obstructions to convexity exist. While local obstructions have been studied extensively, non-local obstructions have not. We show that algebraic signatures in the canonical form of the neural ideal reveal these non-local obstructions. We also study the neural ideals of Sunflowers, S_n, and provide closed convex realizations for S_n.

1. Introduction

A neural codeword is the collection of neurons that fire when an animal enters a specific region of space. We index the group of neurons using the set $[n] = \{1, \ldots n\}$. The codeword can be viewed as either an n-long string of binary or the set of positions in the string where the neurons are firing, indicated by a "1". The environment that triggers these neurons to fire is called the stimulus space X.

Definition 1.1. *A* ***neural code****, C, is a set of codewords, which are binary strings of length n.*

Example 1.2. *An example of a neural code on 5 neurons is*

$$C = \{125, 234, 145, 123, 4, 23, 15, 12, \emptyset\}$$

We can write the same code using the binary notation

$$C = \{11001, 01110, 10011, 11100, 00010, 01100, 10001, 11000, 00000\}$$

Definition 1.3. *A* ***receptive field*** *$U_i \subset \mathbb{R}^d$ is the region of space in which the neuron i fires.*

Date: July 26, 2020.

Several receptive fields can overlap. Where these fields overlap, some subset $\sigma \subset [n]$ of the neurons fires. This region is labeled as

$$U_\sigma = \left(\bigcap_{i\in\sigma} U_i\right) \setminus \left(\bigcup_{j\notin\sigma} U_j\right).$$

The collection of receptive fields $U = \{U_i\}_{i\in[n]}$ cover the stimulus space X. We can associate a code to this cover.

Definition 1.4. *Let X be a stimulus space. Given an open cover of X, $U = \{U_1, \ldots, U_n\}$, we define the associated code $C(U)$ as follows:*

$$C(U) \stackrel{def}{=} \left\{\sigma \subset [n] \mid \left(\bigcap_{i\in\sigma} U_i\right) \Big/ \left(\bigcup_{j\in[n]/\sigma} U_j\right) \neq \emptyset\right\}$$

Definition 1.5. *A cover U is called a **realization of** C if $C = C(U)$. If all of these sets are open (closed) and convex, we refer to the realization as being open (closed) convex. We also describe C as open (closed) convex.*

Definition 1.6. *For $\sigma, \tau \subset [n]$ with $\sigma \neq \emptyset$ and $\sigma \cap \tau = \emptyset$, we say that (σ, τ) is a **receptive field (RF) relationship** of a code C if*

$$U_\sigma \subset \textstyle\bigcup_{i\in\tau} U_i \text{ and } U_\sigma \cap U_i \neq \emptyset \text{ for all } i \in \tau,$$

for any $U = \{U_1, \ldots, U_n\}$ where $C = C(U)$.

Definition 1.7. *A **minimal** RF relationship (σ, τ) requires all neurons in σ or τ for the containment $U_\sigma \subset \bigcup_{i\in\tau} U_i$ to hold.*

Several obstructions to convexity have been discovered. These obstructions are aspects of the code that prevent a convex realization from being possible. One such obstruction is a local obstruction. The definition of a local obstruction relies on the definition of a simplicial complex and the link of a codeword within that complex.

Definition 1.8. *The **simplicial complex of the code** $\Delta(C)$ is the smallest abstract simplicial complex on the index set such that $C \subset \Delta(C)$.*

Definition 1.9. *Let $\Delta(C)$ be the simplicial complex for a code. For any $\sigma \in \Delta(C)$, the **link of** σ **in** Δ is*

$$Lk_\sigma(\Delta) \stackrel{def}{=} \{\omega \in \Delta \mid \sigma \cap \omega = \emptyset \text{ and } \sigma \cup \omega \in \Delta\}$$

Definition 1.10. *A topological space Y is termed **contractible** if there exists continuous maps $f : Y \to \{*\}$ and $g : \{*\} \to Y$ such that*

$g \circ f \sim id_{\{*\}}$ and $g \circ f \sim id_Y$, i.e., the space can be continuously deformed to a point.

The link gives information about those U_j which overlap U_σ, for which $j \notin \sigma$. A local obstruction occurs when the link of a codeword is not contractible and that codeword is not in the code. It has been shown that no convex code can have a local obstruction. And, various means for detecting the presence of the local obstruction have been found. We examine a generating set of a certain ideal, called the neural ideal.

Definition 1.11. *The **neural ideal of the code** is an ideal of the polynomial ring $\mathbb{F}_2[x_1, \ldots, x_n]$ and consists of all polynomials whose zeros are precisely the codewords in C.*

$$J_C =< \chi_\nu \mid \nu \in \mathbb{F}_2^n \setminus C >,$$

$$\chi_\nu = \prod_{i|\nu_i=1} x_i \prod_{j|\nu_j=0} (1 + x_j)$$

The generating set of interest is called the canonical form, which consists of the set generated by the minimal pseudo-monomials in the neural ideal.

Definition 1.12. *A **pseudo-monomial** is a polynomial with the form*

$$\chi = \textstyle\prod_{i\in\mu} x_i \prod_{j\in\tau}(1 + x_j) \text{ for } \mu, \tau \subset \{1, \ldots, n\} \text{ and } \mu \cap \tau = \emptyset.$$

*A pseudo-monomial χ_{ν_1} is **minimal** in J_C if no other pseudo-monomial χ_{ν_2} in J_C divides χ_{ν_1}.*

Definition 1.13. *The **Canonical Form** of J_C is*

$$CF(J_C) = \{minimal\, pseudo\text{–}monomials\, of\, J_C\}$$

Being made up of the minimal pseudo-monomials, the canonical form of the neural ideal indicates those receptive field relationships which must occur in any realization of the code. The relationships implied by the pseudo-monomials have been studied and proven in [1]. We list some of the relationships that will be pertinent to our results.

(1) $x_i x_j x_k \in CF \implies U_{ijk} = \emptyset$,
(2) $x_i x_j (x_k + 1) \in CF \implies U_{ijk} \neq \emptyset$
(3) $x_i(x_j + 1)(x_k + 1) \in CF \implies U_i \subset (U_j \cup U_k)$.

Definition 1.14. *An **algebraic signature** for some property is a subset of an algebraic set that encodes the property in question.*

Even if no local obstructions are present, a code can be non-convex. An example of a non-open-convex code having no local obstructions was seen in [2]. This code does have a closed convex realization. Likewise, three other codes were found to be absent of local obstructions, open convex, but not closed-convex. These three codes were proven to be non-closed convex in [3]. We develop a criterion for an algebraic signature that shows this obstruction to non-closed convexity. This is the first criterion for non-closed-convexity, besides local obstructions.

2. Main Results

2.1. **A Non-local Obstruction to Closed Convexity.** Our main result about non-local obstructions to closed convexity is

Theorem 1. *Let C be a code on n neurons. Let $i, j, k, l, m \in [n]$. Suppose the canonical form of the neural ideal of C has the following subset of pseudo-monomials:*

$$\{x_i x_k(x_j + 1), x_j x_m(x_i + 1), x_k x_m(x_l + 1), x_m x_k x_i,$$
$$x_m x_k x_j, x_k(x_l + 1)(x_j + 1), x_j(x_i + 1)(x_k + 1)\}$$

Then, the code C is not closed convex.

Before proving the theorem, we will prove three lemmas that we will use in the proof of the theorem. The proofs for the lemmas and the theorem abstract the proofs used by Cruz et al. [4] and Goldrup and Phillipson [3].

Lemma 2.1. *Let C be a convex neural code. If $x_i x_k x_m, x_i x_k(x_j + 1), x_j x_m(x_i + 1), x_k x_m(x_l + 1) \in CF(J_C)$, then the sets U_{ijk}, U_{ijm}, and U_{klm} are nonempty and disjoint; and any three points $y_{ijk} \in U_{ijk}$, $y_{ijm} \in U_{ijm}$, and $y_{klm} \in U_{klm}$ are not colinear.*

Proof. The fact that $x_i x_k(x_j+1), x_j x_m(x_i+1), x_k x_m(x_l+1)$ imply that $U_{ijk} \neq \emptyset$,$U_{ijm} \neq \emptyset$, and $U_{klm} \neq \emptyset$, respectively (Lemma 4.2 [1]). We claim that the sets U_{ijk}, U_{ijm}, and U_{klm} are pairwise disjoint. We will prove it for just U_{ijk} and U_{ijm}. We know that if $x_i x_k x_m \in CF(J_C)$, then $U_{ikm} = \emptyset$ $(*)$. Hence, $U_{ik} \cap U_{im} = \emptyset$. Since $U_{ijk} \subset U_{ik}$ and $U_{ijm} \subset U_{im}$, we have $U_{ijk} \cap U_{ijm} = \emptyset$.

Since the sets are disjoint, we can find three distinct points in the sets:

$$y_{ijk} \in U_{ijk},\ y_{ijm} \in U_{ijm}, \text{ and } y_{klm} \in U_{klm}$$

We draw a line between each pair of points. Consider the line $L_1 = \overline{y_{ijm}y_{ijk}} \subset U_ij$. By $(*)$, $U_{ij} \cap U_{km} = \emptyset$, and thus $U_{ij} \cap U_{klm} = \emptyset$. Hence, $L_1 \cap U_{klm} = \emptyset$.

Consider the line $L_2 = \overline{y_{ijm}y_{ijk}} \subset U_{ij}$. By $(*)$, $U_{ij} \cap U_{km} = \emptyset$, and thus $U_{ij} \cap U_{klm} = \emptyset$. Hence, $L_2 \cap U_{klm} = \emptyset$.

Consider the line $L_3 = \overline{y_{ijm}y_{klm}} \subset U_m$. By $(*)$, $U_m \cap U_{ik} = \emptyset$, and thus $U_m \cap U_{ijk} = \emptyset$. Hence, $L_3 \cap U_{klm} = \emptyset$.

Therefore, any three points $y_{ijk} \in U_{ijk}$, $y_{ijm} \in U_{ijm}$, and $y_{klm} \in U_{klm}$ are not colinear cannot be colinear. See Fig. 1. $\square$

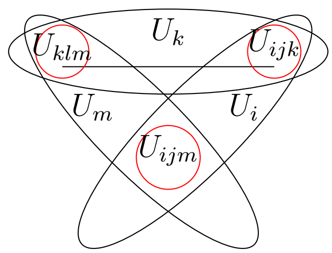


Figure 1. The receptive field structure described in Lemma 2.1. Not having a triple-wise intersection forces any two pair-wise intersections to be disjoint and points in all three pair-wise intersections to be non-colinear,

Lemma 2.2. *Let C be a convex neural code with a realization $U = \{U_\alpha\}_{\alpha=1}^n$. Let $i, j, k \in [n]$. If $x_j(x_i + 1)(x_k + 1) \in CF(J_C)$, then any line segment with an endpoint $y_{ij} \in U_{ij}$ and an endpoint in $y_{jk} \in U_{jk}$ passes through the nonempty intersection U_{ijk}.*

Proof. Let $L = \overline{y_{ij}y_{jk}}$. Then, $L \subset U_j$. Because $x_j(x_i + 1)(x_k + 1) \in CF(J_C)$, we have $U_j \subset (U_i \cup U_k)$. Thus, the sets U_i and U_k cover U_j and, hence, cover L. Since $L = \overline{y_{ij}y_{jk}}$, $L \cap U_i \neq \emptyset$ and $L \cap U_k \neq \emptyset$. Being connected, $L \cap U_{jkl} \neq \emptyset$. See Fig. 2. $\square$

Lemma 2.3. *Let C be a convex neural code with realization $U = \{U_\alpha\}_{\alpha=1}^n$. Let $i, j, k, l, m \in [n]$ If $x_i x_k x_m \in CF(J_C)$, then any line segment with endpoints $y_{ijk} \in U_{ijk}$ and $y_{ijm} \in U_{ijm}$ must contain a point $y_{ij} \in U_{ij} \setminus (U_{ijk} \cup U_{ijm})$.*

Proof. The presence of $x_i x_k x_m \in CF(J_C)$ implies that $U_k \cap U_{im} = \emptyset$ and thus, $U_{ijk} \cap U_{ijm} = \emptyset$, letting us find distinct points $y_{ijk} \in U_{ijk}$ and $y_{ijm} \in U_{ijm}$. Let $L = \overline{y_{ijk}y_{ijm}}$. Because $L \subset U_{ij}$, $U_{ijk} \cap U_{ijm} = \emptyset$, and L is connect, L passes through $U_{ij} \setminus (U_{ijk} \cup U_{ijm})$. Therefore, there exists $y_{ij} \in U_{ij} \setminus (U_{ijk} \cup U_{ijm})$. See Fig. 3. $\square$

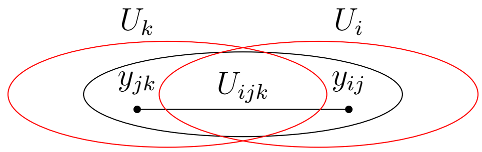


FIGURE 2. The receptive field structure described in Lemma 2.2. How the presence of $x_j(x_i + 1)(x_k + 1) \in CF(J_C)$ causes lines to be covered in convex realizations of the code.

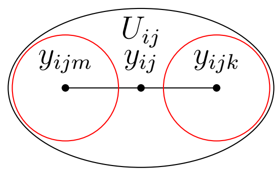


FIGURE 3. The receptive field structure described in Lemma 2.3. How lines contained within a convex receptive field act when their endpoints are in disjoint intersections of receptive fields

We now prove the theorem.

Proof. Assume for contradiction that U is a closed convex realization of C. By Lemma 2.1, there exists a non-degenerate triangle with vertices $y_{ijk} \in U_{ijk}$, $y_{ijm} \in U_{ijm}$, and $y_{klm} \in U_{klm}$. For reference, see Fig. 4. With the assumption that U is closed, we assume that y_{ijk} is the closest point in U_{ijk} to the line $L_3 = \overline{y_{klm}y_{ijm}}$. We label the other sides of the triangle as $L_1 = \overline{y_{ijk}y_{ijm}}$ and $L_2 = \overline{y_{ijk}y_{klm}}$.

After building this triangle, we use Lemma 2.2 and the polynomial $x_k(x_l + 1)(x_j + 1)$ to find a point $y_{jkl} \in U_{jkl}$ on L_2. We use Lemma 2.3 and the pseudo-monomial $x_m x_k x_i$ to get a point $y_{ij} \in L_1 \setminus (U_k \cup U_m)$ on L_1.

We draw $M = \overline{y_{ij}y_{jkl}}$. Lemma 2.2 and the pseudo-monomial $x_j(x_i + 1)(x_k + 1)$ allow us to obtain a point $y'_{ijk} \in M$. Since this point is in the interior of the triangle, it is closer to L_3 than y_{ijk}, a contradiction. Therefore, C is not closed convex. $\square$

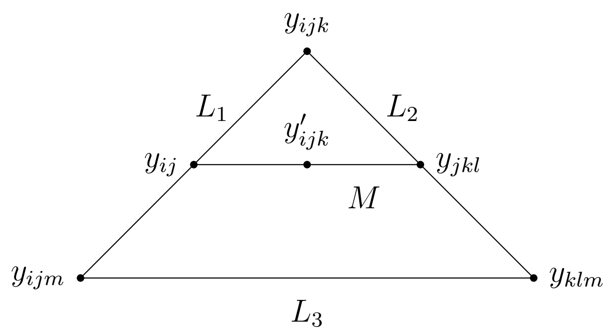


FIGURE 4. Receptive field structure that prevents the code from being closed convex

Corollary 2.4. *Suppose a code C satisfies the following*

(1) *The code contains the codeword with ijk, a codeword with ijm, and a codeword with klm*
(2) *No codewords contain ikm or jkm*
(3) *Every codeword that contains k also contains j or l*
(4) *No codewords that contains j also contains i or k*

Then, C is not closed covex.

Proof. Suppose that the code satisfies the listed properties. From the first property, we know that the intersections U_{ijk}, U_{ijm}, and U_{klm} are nonempty. If no codeword containing ikm or jkm is in C, then $x_m x_k x_i$, $x_m x_k x_i \in CF(J_C)$. If no codewords contain k but not j or l, then $x_k(x_l+1)(x_j+1) \in CF(J_C)$, and if no codeword in C contains j but not i or k, then $x_j(x_i+1)(x_k+1) \in CF(J_C)$, and the result follows. $\square$

Example 2.5. *Goldrup and Phillipson proved that the following three codes are open convex but not closed convex [3]:*

$$C6 = \{125, 234, 145, 123, 4, 23, 15, 12, \emptyset\}$$
$$C10 = \{134, 245, 234, 135, 12, 1, 5, 34, 13, 2, 24, \emptyset\}$$
$$C15 = \{145, 125, 123, 234, 345, 23, 15, 45, 34, 12, \emptyset\}$$

The algebraic signatures for these three codes are:

$$\begin{aligned}
C6 :&(i,j,k,l,m) \to (3,2,1,5,4)\\
&\{x_1x_3(x_4+1), x_1x_4(x_5+1), x_2x_4(x_3+1), x_4x_3x_1, x_4x_2x_1,\\
&\ x_1(x_2+1)(x_5+1), x_2(x_1+1)(x_3+1)\}\\
C10 :&(i,j,k,l,m) \to (1,3,4,2,5)\\
&\{x_1x_4(x_3+1), x_3x_5(x_1+1), x_4x_5(x_2+1), x_5x_4x_1, x_5x_4x_3,\\
&\ x_4(x_2+1)(x_3+1), x_3(x_1+1)(x_4+1)\}\\
C15 :&(i,j,k,l,m) \to (1,2,3,4,5)\\
&\{x_1x_3(x_2+1), x_2x_5(x_1+1), x_3x_5(x_4+1), x_5x_3x_1, x_5x_3x_2,\\
&\ x_3(x_4+1)(x_2+1), x_2(x_1+1)(x_3+1)\}
\end{aligned}$$

2.2. **Algebraic Signatures of Sunflower Codes.** The following definition was the definition for the sunflower codes provided by Jeffs in [5].

Definition 2.6. *Let $n \geq 2$. Define the sunflower code, $S_n \subset 2^{[2n+2]}$, to be the combinatorial code that consists of the following codewords:*

(1) $\emptyset$,
(2) *All codewords of the form $\sigma(n+1)$ for σ a nonempty proper subset of $[n]$,*
(3) $n+1+j$ *for* $1 \leq j \leq n+1$,
(4) $(1\cdots(i-1)(i+1)\cdots n)(n+1)(n+1+i)$ *for* $1 \leq i \leq n$,
(5) *the codeword* $1\cdots n(n+1)(2n+2)$, *and*
(6) *the codeword* $(n+2)(n+3)\cdots(2n+2)$.

Example 2.7.

$$\begin{aligned}
S_2 &= \{\emptyset, 4, 5, 6, 13, 23, 234, 135, 1236, 456\}\\
S_3 &= \{\emptyset, 5, 6, 7, 8, 14, 24, 124, 34, 134, 234, 2345, 1247, 1346, 12348, 5678\}\\
S_4 &= \{\emptyset, 6, 7, 8, 9, 10, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 1245, 345,\\
&\quad 1345, 2345, 13457, 12359, 12458, 23456, 12345(10), 6789(10)\}
\end{aligned}$$

We view the definition through the viewpoint that the "face", F, of the flower is made up of all of the subsets of $[n]$. We then add $(n+1)$-st neuron to all of these subsets. The petals come out from the $(n-1)$ faces of F and meet at the center of the sunflower. These petals are U_{n+1+j} for $1 \leq j \leq n+1$. The center is $(n+2)(n+3)\cdots(2n+2)$.

A Sage algorithm for computing these codes is shown below along with an example of using the algorithm to compute S_2.

```
def sunflower(n):
    A=[]
    H=[]
    for a in powerset(range(1,n+1)):
        a.append(n+1)
        A.append(a)
        B=A[1:2^n-1]
    for g in B:
        h=''.join(map(str,g))
        H.append(h)
    C=[]
    E=[]
    for c in range(1,n+2):
        C.append([n+1+c])
    for d in C:
        e=''.join(map(str,d))
        E.append(e)
    L=range(1,n+1)
    M=Combinations(L, n-1).list()
    N=[]
    for i in range(1,n+1):
        m=M[n-i]
        m.append(n+1)
        m.append(n+1+i)
        p=''.join(map(str,m))
        N.append(p)
    W=range(1,n+1)
    W.append(n+1)
    W.append(2*n+2)
    X=[''.join(map(str,W))]
    Y=[]
    for j in range(n+2,2*n+3):
        Y.append(j)
        Z=[''.join(map(str,Y))]
    return E+H+N+X+Y

sunflower(2)

['4', '5', '6', '13', '23', '234',
'135', '1236', '456']
```

Lemma 2.8. *Let C be a neural code on $n+1$ neurons with realization $U = \{U_\alpha\}_{\alpha\in[n+1]}$. Then, $\bigcup_{i\in[n]} U_i = U_{n+1}$ if and only if $\{x_{n+1}\prod_{i\in[n]}(x_i+1), x_1(x_{n+1}+1), \dots, x_1(x_{n+1}+1)\} \subset CF(J_C)$.*

Proof. For the forward direction, we assume that $\bigcup_{i\in[n]} U_i = U_{n+1}$. We need to prove that both $(1\cdots n, n+1)$ and $(i, n+1)$ for $i \in [n]$ form minimal RF relationships of the code C.

By assumption, $U_{n+1} \subset \bigcup_{i\in[n]} U_i$, one of the requirements for the RF relationship. Because $\bigcup_{i\in[n]} U_i \subset U_{n+1}$, we have $U_{n+1} \cap U_i \neq \emptyset$ for $i \in [n]$. Hence, $(1\cdots n, n+1)$ is a RF relationship. Since we cannot remove $n+1$ without ruining the containment, the RF relationship is minimal. We know

$$\begin{gathered}(1\cdots n, n+1) \text{ is a minimal RF relationship } \iff \\ x_{n+1}\prod_{i\in[n]}(x_i+1) \in CF(J_C) \text{ (Lemma 1.4 [6])}\end{gathered}$$

Now, we prove that $(i, n+1)$ is a minimal RF relationship for $i \in [n]$. By assumption, $\bigcup_{i\in[n]} U_i \subset U_{n+1}$. Thus, $U_i \subset U_{n+1}$ and $U_i \cap U_{n+1} \neq \emptyset$. Hence, $(i, n+1)$ is a RF relationship. Since we cannot remove $n+1$ without ruining the containment, the RF relationship is minimal. We know

$$\begin{gathered}(i, n+1) \text{ being a minimal RF relationship } \iff \\ x_i(x_{n+1}+1) \in CF(J_C) \text{ (Lemma 1.4 [6])}\end{gathered}$$

Therefore, $\{x_{n+1}\prod_{i\in[n]}(x_i+1), x_1(x_{n+1}+1), \dots, x_1(x_{n+1}+1)\} \subset CF(J_C)$.

On the contrary, we assume that $\{x_{n+1}\prod_{i\in[n]}(x_i+1), x_1(x_{n+1}+1), \dots, x_1(x_{n+1}+1)\} \subset CF(J_C)$. We have

$$\begin{gathered}(1\cdots n, n+1) \text{ is a minimal RF relationship } \iff \\ x_{n+1}\prod_{i\in[n]}(x_i+1) \in CF(J_C) \text{ (Lemma 1.4 [6])}\end{gathered}$$

and

$$\begin{gathered}(i, n+1) \text{ being a minimal RF relationship } \iff \\ x_i(x_{n+1}+1) \in CF(J_C) \text{ (Lemma 1.4 [6]).}\end{gathered}$$

From the first, we have $U_{n+1} \subset \bigcup_{i\in[n]} U_i$. From the second, we have that $U_i \subset U_{n+1}$ for all $i \in [n]$. Thus, $U_{n+1} = \bigcup_{i\in[n]} U_i$ □

Corollary 2.9. *Let C be a neural code on $n+1$ neurons with realization $U = \{U_\alpha\}_{\alpha\in[n+1]}$. Let $\sigma \in C$ be nonempty. The codeword σ contains $n+1$ and $n+1 \notin C$ if and only if $\{x_{n+1}\prod_{i\in[n]}(x_i+1), x_1(x_{n+1}+1), \dots, x_1(x_{n+1}+1)\} \subset CF(J_C)$.*

Proof. We prove the statement

For all nonempty $\sigma \in C$, σ contains $n+1$ and $\sigma \neq n+1 \iff \bigcup_{i\in[n]} U_i = U_{n+1}$

Suppose that, for all nonempty $\sigma \in C$, σ contains $n+1$ and $\sigma \neq n+1$. Then, $\bigcup_{\sigma\in C} U_\sigma \subset U_{n+1}$. Hence, $\bigcup_{i\in[n]} U_i \subset U_{n+1}$. On the other hand, $n+1 \notin C$. Thus, $U_{n+1} \subset \bigcup_{\sigma\in C} U_\sigma$. Therefore, $\bigcup_{i\in[n]} U_i = U_{n+1}$.

On the contrary, suppose $\bigcup_{i\in[n]} U_i = U_{n+1}$. Then, $U_i \subset U_{n+1}$ for all $i \in [n]$. Then, for any $\sigma \in C$, $U_\sigma \subset U_{n+1}$. Thus, $(n+1) \in \sigma$. Since $U_{n+1} \subset \bigcup_{i\in[n]} U_i$, $n+1 \notin C$. □

Lemma 2.10. *Let C be a code on n neurons. Let $1 \leq i < j < k \leq n$. Let $U = \{U_\alpha\}_{\alpha=1}^n$ be a realization of C. Then,*

$$U_i \cap U_j = U_j \cap U_k = U_i \cap U_k = U_{ijk} \neq \emptyset$$

$$\Updownarrow$$

$$\{x_i x_j (x_k+1), x_i(x_j+1)x_k, (x_i+1)x_j x_k\} \subset CF(J_C)$$

Proof. We prove the forward direction first. Suppose

$$U_i \cap U_j = U_j \cap U_k = U_i \cap U_k = U_{ijk} \neq \emptyset.$$

Then, $U_i \cap U_j \subset U_k$, $U_j \cap U_k \subset U_i$, and $U_i \cap U_k \subset U_j$. Hence, (ij,k),(jk,i), and (ik,j) are RF relationships. They are minimal, because we cannot remove k, j, or i from the respective containments without contradicting the assumption that the intersections are not empty.

(ij,k) being a minimal RF relationship $\iff x_i x_j(x_k+1) \in CF(J_C)$ (Lemma 1.4 [6])

Because this happens for the other two RF relationships, $\{x_i x_j(x_k+1), x_i(x_j+1)x_k, (x_i+1)x_j x_k\} \subset CF(J_C)$

To prove the other direction, we assume $\{x_i x_j(x_k+1), x_i(x_j+1)x_k, (x_i+1)x_j x_k\} \subset CF(J_C)$. We know that

$x_i x_j(x_k+1) \in CF(J_C) \iff (ij,k)$ is a minimal RF relationship (Lemma 1.4 [6])

This statement implies that $U_i \cap U_j \subset U_k$, $U_j \cap U_k \subset U_i$, and $U_i \cap U_k \subset U_j$. Therefore, $U_i \cap U_j = U_j \cap U_k = U_i \cap U_k = U_{ijk} \neq \emptyset$ □

When $n = 3$, this situation is what Jeffs calls a sunflower of convex open sets with three petals in $\mathbb{R}^2$ [5].

Lemma 2.11. *Let C be a neural code on n neurons with realization $U = \{U_\alpha\}_{\alpha=1}^n$. Let $i \in [n]$. The codeword $i \in C$ if and only if $x_i \prod_{j\in\tau}(x_j+1) \notin CF(J_C)$ for $\tau \subset [n] \setminus \{i\}$, where $\tau \neq \emptyset$.*

Proof. We prove the forward direction first. Suppose $i \in C$. For the sake of contradiction, we assume that $x_i \prod_{j\in\tau}(x_j + 1) \in CF(J_C)$ for some $\tau \subset [n] \setminus \{i\}$. Then, (i, τ) form a minimimal RF relationship. Because (i, τ) is a RF relationship, $U_i \subset \bigcup_{j\in\tau} U_j$. Thus, $\bigcup_{j\in\tau} U_j$ covers U_i. Hence, i only co-fires with some neuron in τ. Therefore, $i \notin C$.

On the contrary, suppose that $x_i \prod_{j\in\tau}(x_j + 1) \notin CF(J_C)$ for $\tau \subset [n] \setminus \{i\}$. Then, $U_i \not\subset \bigcup_{j\in\tau} U_j$ for any $\tau \subset [n] \setminus \{i\}$. Thus, U_i is not covered by any collection of the other receptive fields of U. Therefore, $i \in C$. □

Theorem 2. *The algebraic signature for the sunflower code S_n has the following properties.*

(1) $\{x_i x_j (x_k + 1), x_i(x_j + 1)x_k, (x_i + 1)x_j x_k\} \subset AS(S_n)$ *for* $i, j, k \in \{n + 2, n + 3, \dots, 2n + 2\}$
(2) $x_i(x_{n+1} + 1) \in AS(S_n)$ *for* $i \in [n]$ *and* $x_{n+1} \prod_{j\in[n]}(x_j + 1) \in AS(S_n)$.
(3) $x_i \prod_{j\in\tau}(x_j + 1) \notin AS(S_n)$ *for* $i \in \{n + 2, \dots, 2n + 2\}$ *and* $\tau \subset [n + 1]$
(4) $(x_{2n+2} + 1)x_{(n+1)}x_n \cdots x_1 \in AS(S_n)$

Proof. The first three conditions follow, respectively from Lemmas 2.8, 2.9, and 2.10. We prove the fourth condition here.

We need to prove that $(\sigma, 2n + 2)$, where $\sigma = 1 \cdots n(n + 1)$, forms a minimal RF relationship of the code C. By definition of the code, $U_\sigma \subset U_{2n+2}$. Thus, $(\sigma, 2n + 2)$ is a receptive field relationship. If we were to remove $2n + 2$ from this containment, then $U_\sigma = \emptyset$. Thus, $(\sigma, 2n + 2)$ is minimal. We know that

$$(\sigma, 2n + 2) \text{ being a minimal receptive field relationship } \iff x_\sigma(x_{2n+2} + 1) \in CF(J_C) \text{ ([6])}$$

Therefore, $x_\sigma(x_{2n+2} + 1) \in CF(S_n)$. □

Conjecture 2.12. *The algebraic signature for the sunflower code S_n contains the following type of pseudo-monomial.*

$x_i x_j (x_k + 1)$ *for* $i \in \{n+2, \dots, 2n+2\}$, $j, k \in ([n] \setminus \{i\} \cup \{n+1\} \cup \{i\})$

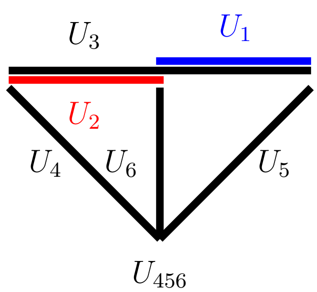
These pseudo-monomials are believed to encode the fact that the codewords $n + 1 + i$ for $1 \leq i \leq n$ intersect the receptive fields $U_{1\cdots(i-1)(1+1)\cdots n(n+1)}$.

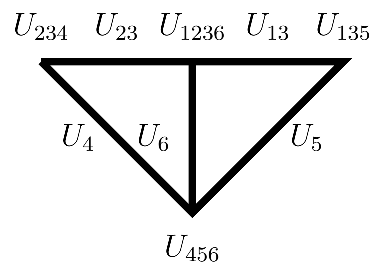
2.3. **Closed convexity of sunflower codes.** Although the sunflower codes are not open convex, they are closed convex.

Theorem 3 (Closed convexity of sunflowers)**.** *The sunflower code S_2 is closed convex in $\mathbb{R}^2$. The sunflower code S_n, $n \geq 3$, is closed convex in $\mathbb{R}^3$.*

Proof. The realization for S_2 is shown in Fig. 2.

$$S_2 = \{\emptyset, 4, 5, 6, 13, 23, 234, 135, 1236, 456\}$$





(A)

FIGURE 5. Receptive field setup (a) and realization (b) of S_2

The realization for $n \geq 3$ is drawn as follows

(1) Draw a $(2^n - 2)$-sided, regular polygon. This polygon is the receptive field for the codeword $1 \cdots n(n+1)(2n+2)$.
(2) Draw the circle that passes through the vertices of the polygon. The circle is U_{n+1}. The clopen subset of the circle outside of one of the edges of the polygon corresponds to one of the nonempty proper subsets of $[n]$.
(3) Pick a point in a plane parallel to the one in which the polygon sits and let $U_{2n+2} = conv\{point, vertices\}$.
(4) Draw a line segment from each subset of the circle $U_{1\cdots(i-1)(i+1)(n+1)}$ for $1 \leq i \leq n$ to the point from (3). This line is U_{n+1+i}.

Suppose we have one of these realizations. Within the polygon, the codeword for neurons firing is $1 \cdots n(n+1)(2n+2)$. Any region inside of the circle but outside the polygon, the place cells that fire are those that are indexed by $\sigma(n+1)$, where σ is a nonempty proper subset of $[n]$. Within these regions in the circle but outside of the polygon, there are regions labeled as $(1 \cdots (i-1)(i+1) \cdots n)(n+1)$ for $1 \leq i \leq n$. The region either fires $(1 \cdots (i-1)(i+1) \cdots n)(n+1)$ or $(1 \cdots n)(n+1)(n+1+i)$, the latter firing where the line $n+1+i$ intersects the region. In the convex hull between the points but outside of the polygon and point p fires the $2n+2$ neuron. At p, the neurons firing are $(n+2)(n+3)\cdots(2n+2)$. Outside of these regions, no neurons indexed by $[n]$ fire. The code generated by this realization is then

(1) $1 \cdots n(n+1)(2n+2)$
(2) $\sigma(n+1)$, for σ a proper, nonempty subset of $[n]$.
(3) $(1 \cdots (i-1)(i+1) \cdots n)(n+1)(n+1+i)$ for $1 \leq i \leq n$
(4) $n+1+i$ $1 \leq i \leq n$
(5) $2n+2$
(6) $(n+2)(n+3) \cdots (2n+2)$
(7) $\emptyset$

This code $C(U) = S_n$. $\square$

Examples for making the face of S_3 and S_4 are shown in Fig. 6 and Fig. 7, respectively.

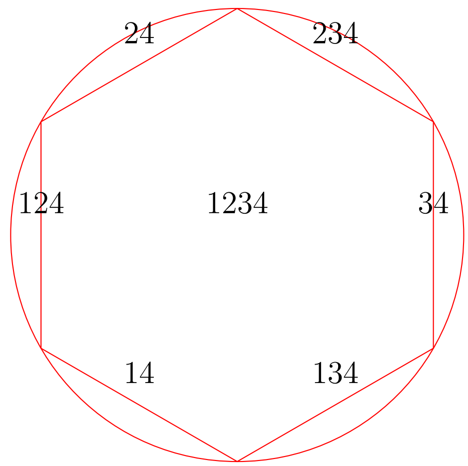


FIGURE 6. Face of S_3

$$S_3 = \{\emptyset, 5, 6, 7, 8, 14, 24, 124, 34, 134, 234,\\ 2345, 1247, 1346, 12348, 5678\}$$

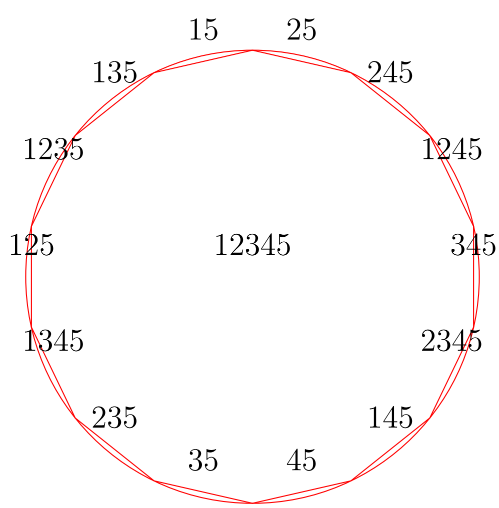


FIGURE 7. Face of S_4 based on L28 from Curto et al. [7]

$$S_4 = \{\emptyset, 6, 7, 8, 9, 10, 15, 25, 125, 35, 135, 235, 1235,\\ 45, 145, 245, 1245, 345, 1345, 2345, 13457, 12359,\\ 12458, 23456, 12345(10), 6789(10)\}$$

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