A Central Element of the Quantum Group $U_q(\mathfrak{so}_{2n})$

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Texas A&M Probability and Algebra REU

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Outline of the talk

- Define basic algebraic structures and research problem
- Apply main formula for simple cases $n = 3, 4$
- Describe additional progress for general n
- Show some probabilistic applications

The underlying Lie algebra

Definition

The Lie algebra $\mathfrak{so}_{2n}(\mathbb{C})$ is the set of $2n \times 2n$ matrices

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A = -D^T, B^T = -B, C^T = -C \right\},$$

where $A, B, C, D \in \mathbb{C}^{n \times n}$.

Main difference from ordinary abstract algebra: use the **Lie bracket** $[A, B] = AB - BA$ instead of multiplication.

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Roots, weights, and representations

- Often study operators by analyzing eigenvalues and eigenspaces.
- Analogously, there are two types of "eigenvalues" we'll consider:
    - Weights (denoted μ or λ) for $2n$-dim. **fundamental** representation,
    - Roots (denoted α_i or $-\alpha_i$) for $(2n)^2$-dim. **adjoint** representation.
- Let L_i be a function which sends a matrix M to the diagonal entry M_{ii}. The weights and roots for $\mathfrak{so}_{2n}$ are

$$\mu = \pm L_i, \quad \alpha = \pm L_i \pm L_j$$

for $1 \leq i < j \leq n$.

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The algebras $U(\mathfrak{so}_{2n})$ and $U_q(\mathfrak{so}_{2n})$

- **Universal enveloping algebra** $U(\mathfrak{so}_{2n})$: "allow multiplication, not just bracket."
  - Generated by E_i, F_i, H_i $(1 \leq i \leq n)$. Example $(n = 2)$:

$$E_1 = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \end{bmatrix}, \quad F_2 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

- Our research studies the Drinfeld–Jimbo **quantum group** $U_q(\mathfrak{so}_{2n})$.
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  - Example of an element:

$$(q^2 + 1)E_1^2 F_1 q^{H_1 - H_2}.$$

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Motivation: the Casimir element

- In $U(\mathfrak{so}_{2n})$ (the symmetric case), the **quadratic Casimir element** is a distinguished element of the center.
- This Casimir element can be procedurally represented as a generator matrix of a Markov process.
- Idea: do something similar with $U_q(\mathfrak{so}_{2n})$ (find a Casimir element, then turn into a generator matrix). Should result in an asymmetric process.

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The main problem and formula

Problem

Find an explicit form for a central element of $U_q(\mathfrak{so}_{2n})$ in terms of the generators E_i, F_i, q^{H_i}.

Recall: in $U(\mathfrak{so}_{2n})$, we find dual elements and compute $\sum_i X_i X^i$.

Proposition (Kuan '16)

For each weight μ, let v_μ be a vector in its weight space. Given weights μ, λ, suppose $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ. If $e^*_{\mu\lambda}$ and $f^*_{\lambda\mu}$ are their q-pairing dual elements, and ρ is half the sum of the positive roots of $\mathfrak{g}$, then

$$\sum_\mu q^{(-2\rho,\mu)} q^{H_{-2\mu}} + \sum_{\mu>\lambda} q^{(\mu-\lambda,\mu)} q^{(-2\rho,\mu)} e^*_{\mu\lambda} q^{H_{-\mu-\lambda}} f^*_{\lambda\mu}$$

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- $(-2\rho,\mu)$ and $(\mu-\lambda,\mu)$ are ordinary dot products, so the corresponding terms are just powers of q.
- q^Hs are products of $q^{\pm H_i}$s, which are also simple to compute.

Thus, suffices to understand how $e^*_{\mu\lambda}$ and $f^*_{\lambda\mu}$ look.

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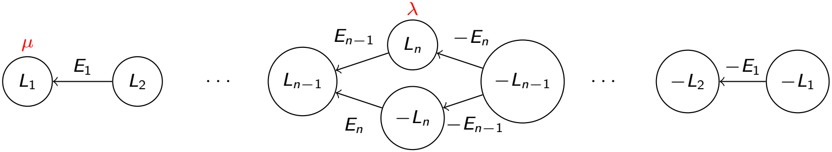
Computing $e_{\mu\lambda}$ and $f_{\lambda\mu}$

The generators E_is and F_is are operators that move us between different weight spaces.



Here, $e_{\mu\lambda}$ and $f_{\lambda\mu}$ send us from λ to μ and vice versa. In this case, $e_{\mu\lambda} = E_1 \cdots E_{n-1}$, and $f_{\lambda\mu} = F_{n-1} \cdots F_1$.

A q-deformed pairing

We introduce a function $\langle\ ,\ \rangle$, which takes in (product of Fs and q^Hs) and (product of Es and q^Hs), outputting (rational function in q). More formally: $U_q(\mathfrak{b}-) \times U_q(\mathfrak{b}+) \to \mathbb{Q}(q)$.

- For the generators, the only nonzero pairings are

$$\langle q^{H_\alpha}, q^{H_\beta}\rangle = q^{-(\alpha\cdot\beta)}, \quad \langle F_i, E_i\rangle = -\frac{1}{q-q^{-1}},$$

  where α and β are linear combinations of the α_is.

- There is also an inductive way to compute things like

$$\langle q^{H_1}F_2F_1, q^{H_2}E_1E_2\rangle,$$

  involving the **coproduct** of the generators.

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Sample values of the pairing

Take $n = 4$. Here are some example computations:

- $\langle F_1 F_2, E_1 E_2 \rangle = \dfrac{1}{(q - q^{-1})^2}$.
- $\langle F_1 F_3 F_3, E_3 E_1 E_3 \rangle = -\dfrac{1}{(q - q^{-1})^3}(q^2 + 1)$.
- $\langle F_1 F_2 F_3, E_1 E_2 E_2 \rangle = 0$.

Lemma (L.)

The q-pairing of a product of Fs and a product of Es is only nonzero if the indices are permutations of each other, in which case it is $(q - q^{-1})^{-n}$ times a Laurent series in q.

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Finding the dual elements

Example: find **dual element under** $\langle\ ,\ \rangle$ of $\underline{F_2F_1}$ for $n = 3$.

- $\{F_1F_2, \underline{F_2F_1}\}$ both have nonzero pairing with both of $\{E_1E_2, E_2E_1\}$. (Call these $\{f_1, \underline{f_2}\}$ and $\{e_1, e_2\}$.)
- **Dual elements** f_i^* are combinations of the e_is, such that $\langle f_i, f_j^* \rangle = \delta_{ij}$.
- Form matrix of pairings M such that $M_{ij} = \langle f_i, e_j \rangle$:

$$M = (q - q^{-1})^2 \begin{bmatrix} 1 & 1/q \\ 1/q & 1 \end{bmatrix}$$

- Invert the matrix and look at corresponding (second) row.

$$M^{-1} = (q - q^{-1}) \begin{bmatrix} q & -1 \\ -1 & q \end{bmatrix}$$

The dual of F_2F_1 is $f^* = \boxed{(q - q^{-1})(-E_1E_2 + qE_2E_1)}$.

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- Matrix M needs to be **invertible**.
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  - Serre relation makes this hard: for example,

$$E_1^2E_2 + E_2E_1^2 = (1+q)E_1E_2E_1$$

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Computational difficulties

Two main reasons this is more complicated than the other steps:

- Matrix M needs to be **invertible**.
  - Need to make sure different f_is and e_is linearly independent
  - Serre relation makes this hard: for example,

$$E_1^2 E_2 + E_2 E_1^2 = (1 + q) E_1 E_2 E_1$$

- Larger sets of indices mean the **dimensions** of M are larger.
  - Inverting symbolic matrices takes time (with a computer)

The central element of $U_q(\mathfrak{so}_6)$

Let $r = q - \frac{1}{q}$, and write (for example) $E_1 E_2 E_3$ as E_{123}.

Theorem (L.)

The following element of the quantum group $U_q(\mathfrak{so}_6)$ is central:

$$\begin{aligned}
& q^{-4-2H_1-H_2-H_3} + q^{-2-H_2-H_3} + q^{H_2-H_3} + q^{H_3-H_2} + q^{2+H_2+H_3} + q^{4+2H_1+H_2+H_3} + \frac{r^2}{q^3} F_1 q^{-H_1-H_2-H_3} E_1 \\
& + \frac{r^2}{q} F_2 q^{-H_3} E_2 - \frac{r^2}{q} F_3 q^{-H_2} E_3 + r^2 q F_2 q^{H_3} E_2 - r^2 q F_3 q^{H_2} E_3 + r^2 q^3 F_1 q^{H_1+H_2+H_3} E_1 \\
& + \frac{r^2}{q^3}(qF_{12} - F_{21}) q^{-H_1-H_3} (qE_{21} - E_{12}) - \frac{r^2}{q^3}(qF_{13} - F_{31}) q^{-H_1-H_2} (qE_{31} - E_{13}) \\
& + r^2 q (qF_{21} - F_{12}) q^{H_1+H_3} (qE_{12} - E_{21}) - r^2 q (qF_{31} - F_{13}) q^{H_1+H_2} (qE_{13} - E_{31}) \\
& - \frac{r^2}{q^3}(q^2 F_{123} - qF_{213} - qF_{312} + F_{231}) q^{-H_1} (q^2 E_{231} - qE_{312} - qE_{213} + E_{123}) \\
& - \frac{r^2}{q}(q^2 F_{231} - qF_{312} - qF_{213} + F_{123}) q^{H_1} (q^2 E_{123} - qE_{213} - qE_{312} + E_{231}) \\
& - \frac{r^4}{q^2}((q^2+1) F_{1231} - qF_{1312} - qF_{2131})((q^2+1) E_{1231} - qE_{1312} - qE_{2131}) \\
& - r^4 F_2 F_3 E_2 E_3 .
\end{aligned}$$

This element acts as a constant $(q^6 + q^2 + 2 + q^{-2} + q^{-6})$ times the identity matrix in the fundamental representation.

Theorem (L.)

The following element of the quantum group $U_q(\mathfrak{so}_8)$ is central:

$$
\begin{aligned}
& q^{-6-2H_1-2H_2-H_3-H_4} + q^{-4-2H_2-H_3-H_4} + q^{-2-H_3-H_4} + q^{H_3-H_4} \\
& + q^{H_4-H_3} + q^{2+H_3+H_4} + q^{4+2H_2+H_3+H_4} + q^{6+2H_1+2H_2+H_3+H_4} \\
& + \frac{r^2}{q^5} F_1 q^{-H_1-2H_2-H_3-H_4} E_1 + \frac{r^2}{q^5}(qF_{12} - F_{21}) q^{-H_1-H_2-H_3-H_4}(qE_{21} - E_{12}) \\
& + \frac{r^2}{q^5}(q^2 F_{123} - qF_{132} - qF_{213} + F_{321}) q^{-H_1-H_2-H_4}(q^2 E_{321} - qE_{213} - qE_{132} + E_{123}) \\
& - \frac{r^2}{q^5}(q^2 F_{124} - qF_{142} - qF_{241} + F_{421} q^{-H_1-H_2-H_3}(q^2 E_{421} - qE_{241} - qE_{142} + E_{124}) \\
& - \frac{r^2}{q^5} \boxed{A_1} q^{-H_1-H_2} \boxed{A_4} - \frac{r^2}{q^5} \boxed{A_5} q^{-H_1} \boxed{A_8} - \frac{r^4}{q^4} \boxed{A_9} \boxed{A_{10}} \\
& + \frac{r^2}{q^3} F_2 q^{-H_2-H_3-H_4} E_2 + \frac{r^2}{q^3}(qF_{23} - F_{32}) q^{-H_2-H_4}(qE_{32} - E_{23}) - \frac{r^2}{q^3}(qF_{24} - F_{42}) q^{-H_2-H_3}(qE_{42} - E_{24}) \\
& - \frac{r^2}{q^3}(q^2 F_{234} - qF_{324} - qF_{423} + F_{432}) q^{-H_2}(q^2 E_{432} - qE_{324} - qE_{423} + E_{234}) \\
& - \frac{r^4}{q^2}((q^2+1)F_{2342} - qF_{3242} - qF_{2423})((q^2+1)E_{2342} - qE_{3242} - qE_{2423}) \\
& - \frac{r^2}{q^3} \boxed{A_7} q^{H_1} \boxed{A_6} + \frac{r^2}{q} F_3 q^{-H_4} E_3 - \frac{r^2}{q} F_4 q^{-H_3} E_4 \cdots
\end{aligned}
$$

The central element of $U_q(\mathfrak{so}_8)$, continued

Theorem

(Here is the rest of the element.)

$$
\begin{aligned}
\cdots - r^4 F_3 F_4 E_4 E_3 - \frac{r^2}{q}(q^2 F_{432} - qF_{324} - qF_{423} + F_{234}) q^{H_2} (q^2 E_{234} - qE_{324} - qE_{423} + E_{432}) \\
- \frac{r^2}{q} \boxed{A_3} q^{H_1+H_2} \boxed{A_2} - r^2 q F_4 q^{H_3} E_4 - r^2 q (qF_{42} - F_{24}) q^{H_2+H_3} (qE_{24} - E_{42}) \\
- r^2 q (q^2 F_{421} - qF_{241} - qF_{142} + F_{124}) q^{H_1+H_2+H_3} (q^2 E_{124} - qE_{142} - qE_{241} + E_{421}) \\
+ r^2 q F_3 q^{H_4} E_3 + r^2 q (qF_{32} - F_{32}) q^{H_2+H_4} (qE_{23} - E_{32}) \\
+ r^2 q (q^2 F_{321} - qF_{213} - qF_{132} + F_{123}) q^{H_1+H_2+H_4} (q^2 E_{123} - qE_{132} - qE_{213} + E_{321}) \\
+ r^2 q^3 F_2 q^{H_2+H_3+H_4} E_2 + r^2 q^3 (qF_{21} - F_{12}) q^{H_1+H_2+H_3+H_4} (qE_{12} - E_{21}) + r^2 q^5 F_1 q^{H_1+2H_2+H_3+H_4} E_1,
\end{aligned}
$$

where the 10 boxed $\boxed{A_i}$s are omitted for brevity. This element acts as $q^8 + q^4 + q^2 + 2 + q^{-2} + q^{-4} + q^{-8}$ times the identity matrix in the fundamental representation.

Dual elements for general n

A strategy for computing certain dual elements:

Proposition (L.)

Suppose each index only shows up once in an element of $e_{\mu\lambda}$ or $f_{\lambda\mu}$. Then the matrix M^{-1} can be inductively computed by tensoring the inverse matrix from a smaller set of indices repeatedly with $\begin{bmatrix} q & -1 \\ -1 & q \end{bmatrix}$.

- Dual of E_1 is $(q-q^{-1})F_1$.
- Dual of E_1E_2 is $(q-q^{-1})(qF_1F_2 - F_2F_1)$.
- Dual of $E_1E_2E_3$ is $(q-q^{-1})(q^2F_1F_2F_3 - qF_1F_3F_2 - qF_2F_1F_3 + F_3F_2F_1)$.

Notably, the dimension (number of rows) of M is always a power of 2.

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Notably, the dimension (number of rows) of M is always a power of 2.

Dual elements for general n, continued

- The above strategy doesn't work for repeated indices.
- However, the dimensions of M for small n show a pattern.
  - The dimension for indices $(2, 2, 3, 4)$ in $n = 4$ is 5.
  - The dimensions for $(1, 2, 2, 3, 4)$ and $(1, 1, 2, 2, 3, 4)$ are 15 and 20.

Conjecture

Suppose the index $x_1 - 1$ is being added to a set of indices $S = (x_1, \cdots, x_m)$ of dimension d, where $x_1 \leq \cdots \leq x_m$ and $x_1 \leq n - 2$.

- If x_1 appears twice and we add $(x_1 - 1)$ once, the dimension becomes $3d$.
- If x_1 appears twice and we add $(x_1 - 1)$ twice, the dimension becomes $4d$.

Suppose M is the a pairing matrix for some basis for S. Then we can find a 3×3 matrix M_1 and a 4×4 matrix M_2, such that the new pairing matrix M' is $M \otimes M_1$ in the first case and $M \otimes M_2$ in the second case.

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From central element to tensor representation

In order to extract the probabilistic interpretation:

- Replace each generator with its **coproduct**. For example,

$$E_i \to E_i \otimes I + q^{H_i} \otimes E_i.$$

  (This is similar to the symmetric case, where $E_i \to E_i \otimes I + I \otimes E_i$.)
- End up with a $4n^2 \times 4n^2$ matrix with coefficients in terms of q.
- In other words, **every single generator** that showed up in the central elements earlier is represented as a $4n^2 \times 4n^2$ matrix.

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Modifying to a generator matrix

The resulting $4n^2 \times 4n^2$ matrix is not yet a generator matrix, just like in the symmetric case.

- Key idea: if $Mv = 0$, where $v = (v_1, \cdots, v_N)$, we can conjugate by a diagonal matrix $D = \mathrm{diag}(v_1, \cdots, v_N)$.
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This procedure differs from the alternative method of subtracting a diagonal matrix and then negating rows.

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Preliminary observations

Recall these properties of the generator matrix in the symmetric case:

- All nonzero off-diagonal entries equal
- $2n$ absorbing states, $2n$ maximal-choice states, all others pairwise.

Similar properties can be observed at least for $U_q(\mathfrak{so}_6)$ and $U_q(\mathfrak{so}_8)$:

- The absorbing and pairwise states interact in the same ways (except the jump rates differ by a factor of q^2, causing **drift**).
- However, only 4 of the $2n$ maximal-choice states are reachable from each other (finite jump rates). No fission or fusion occurs.

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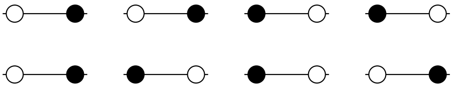
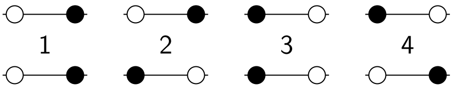


Patterns in the coefficients



Here is the generator submatrix for $n = 3$:

$$\frac{1}{q^6}\begin{bmatrix} -1-2q^2+q^6-q^8-q^{10} & q^2(2-q^4+q^6) & (q^4-1)^2 & q^4(2-q^4+q^6) \\ q^4(2-q^4+q^6) & -1+2q^4+q^8-2q^{10} & 1-q^2+2q^6 & q^2(q^4-1)^2 \\ q^4(q^4-1)^2 & q^2(1-q^2+2q^6) & -q^2-q^4+q^6-2q^{10}-q^{12} & q^4(1-q^2+2q^6) \\ q^6(2-q^4+q^6) & q^2(q^4-1)^2 & q^2(1-q^2+2q^6) & -2q^2+q^4-2q^8-q^{12} \end{bmatrix}$$

- Three different groups: red, blue, green
- Symmetry between q and $\frac{1}{q}$.
- Limit as $q \to 1$.

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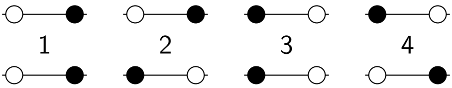

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Thank you!

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