Characterizing Codes with Three Maximal Codewords

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Abstract

Neural codes, represented as collections of binary strings called codewords, are used to encode neural activity. A code is called convex if its codewords are represented as an arrangement of convex sets in Euclidean space. Previous work has focused on addressing the question: how can we tell when a neural code is convex? Giusti and Itskov [3] identified a local obstruction and proved that convex neural codes have no local obstructions. Curto et al. [2] proved the converse for all neural codes on at most four neurons. Lienkaemper et al. [4] provided a counterexample on five neurons to show that this converse is false in general. In fact, unless the maximal codewords intersect in a specific way, convexity is equivalent to being max intersection-complete. In this paper, we prove that the converse holds for all codes with up to three maximal codewords. We also prove results on contractibility, provide convex realizations for all codes, and prove the minimal embedding dimension of such codes is at most two.

1 Introduction

The brain encodes spatial structure through hippocampal neurons known as *place cells*, which are associated with regions of space called *receptive fields*. Place cells fire at a high rate precisely when the animal is in that receptive field. The firing pattern of these neurons form what is known as a neural code. A neural code is convex if it is associated to convex receptive fields. Understanding the convexity of such codes provides insight into how hippocampal neurons affect brain function.

In this paper we focus our attention on characterizing neural codes with few maximal codewords, more specifically codes with three maximal codewords. In the following sections, we define a relationship between codewords, receptive fields, and their convexity. Past work on neural codes with one maximal codeword is widely understood [5] and we provide a proposition for codes with two maximal codewords below. However, such understanding is not complete on neural codes with three maximal codewords.

In this paper, we prove results that completely characterize the convexity of codes with three maximal codewords. First, we prove when links of the resulting simplicial complexes are contractible. Then, we show that the relationship: max intersection-complete $\Leftrightarrow$ convex $\Leftrightarrow$ no local obstructions, holds. Guisti and Itskov [3] identified a local obstruction and proved that convex neural codes have no local obstructions.

In our work, we start by assuming that codes have no local obstructions. Thus, two cases arise. The first case has a non-contractible link of the triplewise intersection of maximal codewords. By the definition of no local obstructions, the code must be max intersection-complete which implies convexity. The second case is when the link of the triplewise intersection is contractible, in which the code doesn't have to be max intersection-complete to imply no local obstructions. Here, we construct a convex realization to prove convexity. Next, we provide both convex open and convex closed realizations for codes with three maximal codewords. Additionally, we prove the minimal embedding dimension is at most two.

In Section 2, we introduce definitions, notation, and previous results. In Section 3, we provide results on contractibility, convexity relationships, convex realizations, and minimal embedding dimensions.

2 Background

In this section, we introduce definitions, notations, and previous results.

2.1 Neural Codes

In a biological context, a codeword represents a set of neurons that fire together while no other neurons fire. A neural code is a set of such codewords, which broadly characterize the combinations of such neurons.

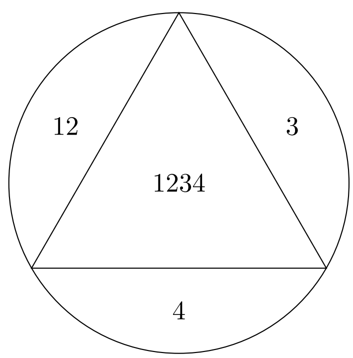
Definition 2.1. A *neural code* $\mathcal{C}$ on n neurons is a set of subsets of $[n]$ (called *codewords*), i.e. $\mathcal{C} \subseteq 2^{[n]}$. A *maximal codeword* (also called a *facet*) in $\mathcal{C}$ is a codeword that is not properly contained in any other codeword in $\mathcal{C}$.

Definition 2.2. For a neural code $\mathcal{C}$ on n neurons, a collection $\mathcal{U} = \{U_1, U_2, ..., U_n\}$ of subsets of a set $\mathcal{X}$ *realizes* $\mathcal{C}$ if a codeword σ is in $\mathcal{C}$ if and only if $(\bigcap_{i\in\sigma} U_i) \setminus \bigcup_{i\notin\sigma} U_i$ is nonempty.

Definition 2.3. A neural code is:

1. *max intersection-complete* if it contains all intersections of maximal codewords in $\mathcal{C}$.
2. a *good-cover code* if it can be realized by a good cover $\mathcal{U} = \{U_1, U_2, ...U_n\}$ of some set $X \subseteq \mathbb{R}^d$.
3. *convex open* if it can be realized by a set of convex open sets $U_1, U_2, ...U_n \subseteq \mathbb{R}^d$. A code's *minimal embedding dimension* is the smallest value of d for which this is possible.
4. *convex closed* if it can be realized by a set of convex closed sets $U_1, U_2, ...U_n \subseteq \mathbb{R}^d$. For closed convex codes, the *minimal embedding dimension* is the minimal d that allows a closed convex realization of $\mathcal{C}$.

Example 2.1. Consider the code $\mathcal{C} = \{\mathbf{1234}, 12, 4, 3, \emptyset\}$, where the maximal codeword is in bold. We interpret this code to mean four neurons fire together in this pattern. There is a region where all four neurons fire together, however, there is not a region for each individual neuron to fire, as neuron 1 and 2 always fire together. A convex realization of this code is seen below:



Now we recall a result of Cruz et al. [1] concerning the minimal embedding dimension for codes that are max intersection-complete:

Lemma 2.1. If $\mathcal{C}$ is a max intersection-complete codes with exactly k maximal codewords, then $\mathcal{C}$ has both convex open and convex closed realizations and the minimal embedding dimension is at most k. The minimal code $\mathcal{C}_{min}$, the smallest code with no local obstructions, can be realized in dimension $k-1$.

2.2 Simplicial Complexes

Definition 2.4. An abstract *simplicial complex* on n vertices in a nonempty set of subsets (*faces*) of $[n]$ that is closed under taking subsets.

For a code $\mathcal{C}$ on n neurons, $\Delta(\mathcal{C})$ is the smallest simplicial complex on $[n]$ that contains $\mathcal{C}$:

$$\Delta(\mathcal{C}) := \{\omega \subseteq [n] \mid \omega \subseteq \sigma \text{ for some } \sigma \in \mathcal{C}\}.$$

Facets are the faces of the simplicial complex that are maximal with respect to inclusion. Note that two codes on n neurons have the same simplicial complex, Δ, if and only if they have the same facets, thus maximal codewords.

Definition 2.5. For a face $\sigma \in \Delta$, the *link* of σ in Δ is the simplicial complex:

$$\mathrm{Lk}_\sigma(\Delta) := \{\omega \subseteq \Delta \mid \sigma \cap \omega = \emptyset, \sigma \cup \omega \in \Delta\}.$$

Example 2.2. The simplicial complex Δ whose facets are $\{1356, 123, 124\}$ and link of the triplewise intersection, $1356 \cap 123 \cap 124 = 1$, are illustrated below:

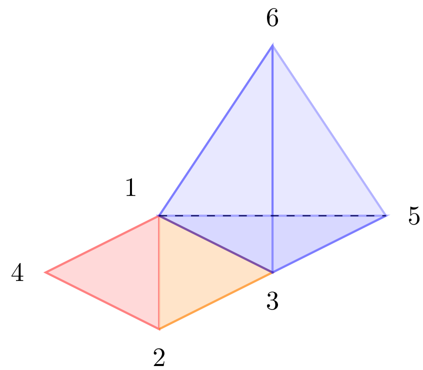


Figure: Δ

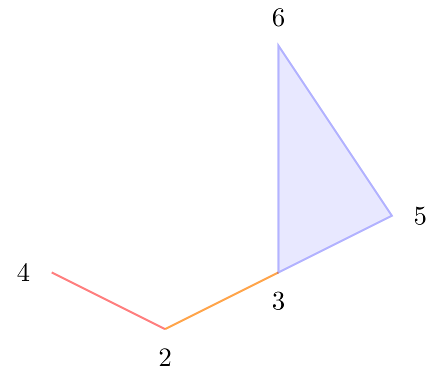


Figure: $Lk_1\Delta$

Definition 2.6. A set is *contractible* if it can be reduced to one of its points by a continuous deformation.

From the figure above, $Lk_1\Delta$ is contractible as it it homotopy-equivalent to a single point.

Now we state a previous result that is important for determining the contractiblity of intersections of maximal codewords.

Lemma 2.2 (Curto et al. [2])**.** Let Δ be a simplicial complex. If $\sigma = \tau_1 \cap \tau_2$, where τ_1, τ_2 are distinct facets of Δ, and σ is not contained in any other facet of Δ, then $Lk_\sigma(\Delta)$ is not contractible.

This result is important for determining the contractibility of codes with three maximal codewords, as we do in Section 3. We only check the links of the triplewise intersections, as the links of the pairwise intersections that are distinct from the triplewise are either represented in the triplewise intersection, or the triplewise intersection is automatically non-contractible. Further on contractibility, we must introduce the definition of nerve, as well as the nerve lemma.

Definition 2.7. For, $\mathcal{W}$, a collection of subsets $\{W_1, W_2, \cdots, W_n\}$ of a set $\mathcal{X}$, the *nerve* of $\mathcal{W}$ is the simplicial complex that records the intersection patterns among the sets:

$$\mathcal{N}(\mathcal{W}) := \{I \subseteq [n] | \textstyle\bigcap_{i \in I} W_i \text{ is nonempty}\}.$$

Lemma 2.3 (Lienkaemper, Shiu, Woodstock, *Lemma 5.1* [4])**.** Let $Lk_\sigma(\Delta)$ where σ is an intersection of facets of the simplicial complex, Δ. Then, we know that $\mathrm{Lk}_\sigma(\Delta) \simeq \mathcal{N}(\mathcal{L}_\sigma(\Delta))$.

This lemma is a strong tool for analyzing the contractibility of codes. In Section 3, we prove the contractilbity of codes using homotopy-equivalences, as proved by Lienkaemper, Shiu, and Woodstock with the nerve lemma.

2.3 Local Obstructions

Definition 2.8. Let $\mathcal{C}$ be a code with simplicial complex $\Delta = \Delta(\mathcal{C})$. A *local obstruction* occurs when the link of the intersection of some maximal codewords in the simplicial complex, $Lk_\sigma(\Delta)$, is not contractible and σ is not in the code.

As defined by Lienkaemper et al. in Definition 2.9 [4]:

Definition 2.9. A face σ of simplicial complex Δ is a *mandatory codeword* if it is the nonempty intersection of a set of facets of Δ such that $Lk_\Delta(\sigma)$ is non-contractible.

Thus, there are no local obstructions if and only if the code contains all mandatory codewords. Previous work has focused on addressing the question: how can we tell a neural code is convex? Giusti and Itskov [3] identified a local obstruction and proved that convex neural codes have no local obstructions.

Furthermore, there is a well known relationship that max intersection-complete $\Rightarrow$ convex $\Rightarrow$ no local obstructions [1]. The first converse is proven to be false in general, and the second converse holds for codes up to two maximal codewords.

Theorem 2.1. Let $\mathcal{C}$ be a neural code with two maximal codewords. Then $\mathcal{C}$ is convex $\Leftrightarrow$ $\mathcal{C}$ has no local obstructions.

Proof. The proof follows easily from the proof of Lemma 2.2. □

The following proposition indicates the complete relationship for codes with two maximal codewords.

Proposition 2.1. Assume $\mathcal{C}$ is a neural code with two maximal codewords. Then convex $\Leftrightarrow$ no local obstructions $\Leftrightarrow$ max intersection-complete.

In order to fully characterize codes with three maximal codewords, this converse must be proven for codes with three maximal codewords.

3 Main Results

In the following sections, we completely characterize codes with three maximal codewords. We provide results on how to determine the contractibility of codes, how to realize such codes convexly, and how the minimal embedding dimensions of these codes change when more non-maximal codewords are added.

Our paper revolves around the known relationship that max intersection-complete $\Rightarrow$ convex $\Rightarrow$ no local obstructions. In the following sections, we prove that max intersection-complete $\Leftrightarrow$ convex (open/closed) $\Leftrightarrow$ no local obstructions.

3.1 Contractibility

In this section we show how to determine the contractibility of the link of the triplewise intersection for neural codes with three maximal codewords. Since we are assuming the code has no local obstructions, if the link of the triplewise intersection is non-contractible, then the code must be max intersection-complete. However, in the case that the link of the triplewise intersection is contractible, we must provide a convex realization to prove that no local obstructions implies convexity. By Lemma 2.2, it is sufficient to only show that the link of the triplewise intersection is non-contractible, as done below.

Theorem 3.1. Let Δ be a simplicial complex with exactly three facets, F_1, F_2, and F_3.

1. If $F_1 \cap F_2 \cap F_3 = \emptyset$ and $F_i \cap F_j \neq \emptyset$ (for some $1 \leq \text{i} < \text{ j} \leq 3$), then $Lk_{(F_i \cap F_j)}\Delta$ is non-contractible.
2. Assume $F_1 \cap F_2 \cap F_3 = \sigma \neq \emptyset$. Let $E_{12} = (F_1 \cap F_2) \setminus F_3$, $E_{13} = (F_1 \cap F_3) \setminus F_2$, and $E_{23} = (F_2 \cap F_3) \setminus F_1$.
   (a) If E_{12}, E_{13}, and $E_{23} = \emptyset$ OR $E_{12} = \alpha \neq \sigma$ and E_{13}, $E_{23} = \emptyset$ or σ, then $Lk_\Delta(F_1 \cap F_2 \cap F_3)$ is non-contractible.
   (b) If $E_{12} = \alpha \neq \sigma$ and $E_{13} = \beta \neq \sigma$ and $E_{23} = \emptyset$ or σ, then $Lk_\Delta(F_1 \cap F_2 \cap F_3)$ is contractible.
   (c) If $E_{12} = \alpha \neq \sigma$, $E_{13} = \beta \neq \sigma$, and $E_{23} = \gamma \neq \sigma$, then $Lk_\Delta(F_1 \cap F_2 \cap F_3)$ is non-contractible.

Statement If $F_1 \cap F_2 \cap F_3 = \emptyset$, then $Lk(F_i \cap F_j)$ is non-contractible. The proof for case 1 in Theorem 3.1 is as follows:

Proof. Suppose $F_1 \cap F_2 \cap F_3 = \emptyset$. Then, the only non-empty intersections are pairwise intersections. By Lemma 2.2, the link of any pairwise intersection, that is not a triplewise intersection, is non-contractible. $\square$

Statement If $F_1 \cap F_2 \cap F_3 \neq \emptyset$ and exactly zero or one pairwise intersection is not equal to the triplewise intersection, then the link of the triplewise intersection is disconnected and thus non-contractible. The proof for case 2a in Theorem 3.1 is as follows:

Proof. Let $F_1 \cap F_2 \cap F_3 = \sigma \neq \emptyset$. There is at most one pairwise intersection of facets that is not equal to the triplewise intersection, so we can select one facet for which all pairwise intersections are equal to the triplewise intersection. Without loss of generality, suppose this facet is F_3. Then, $F_1 \cap F_3 = F_2 \cap F_3 = \sigma$. Consider the link of σ, $\mathrm{Lk}_\sigma(\Delta)$. The facets of $\mathrm{Lk}_\sigma(\Delta)$ are $(F_1 \setminus \sigma), (F_2 \setminus \sigma), (F_3 \setminus \sigma)$. Now, consider the intersections of the facets of the link. We know $(F_1 \setminus \sigma) \cap (F_3 \setminus \sigma) = (F_1 \cap F_3) \setminus \sigma = \sigma \setminus \sigma = \emptyset$. The same holds for $(F_2 \setminus \sigma) \cap (F_3 \setminus \sigma)$. Thus, $(F_3 \setminus \sigma)$ is disjoint from $(F_1 \setminus \sigma)$ and $(F_2 \setminus \sigma)$. Therefore, $\mathrm{Lk}_\sigma(\Delta)$ is disconnected and thus is non-contractible. $\square$

Statement If $F_1 \cap F_2 \cap F_3 \neq \emptyset$ and exactly two pairwise intersections are not equal to the triplewise intersection, then the link of the triplewise intersection is contractible. The proof for case 2b in Theorem 3.1 is as follows:

Proof. Let $F_1 \cap F_2 \cap F_3 = \sigma \neq \emptyset$. Without loss of generality, let the two pairwise intersections distinct from the triplewise intersection be $F_1 \cap F_2 = a, F_2 \cap F_3 = b$. We define the set of facets of $\mathrm{Lk}_\sigma(\Delta)$ as $\mathcal{L}_\Delta(\sigma) = \{(F_1 \setminus \sigma), (F_2 \setminus \sigma), (F_3 \setminus \sigma)\}$. These facets have two pairwise intersections, $a^* = a \setminus \sigma$ and $b^* = b \setminus \sigma$, and all other intersections are empty. Therefore, the nerve, $\mathcal{N}(\mathcal{L}_\Delta(\sigma))$, is as follows:

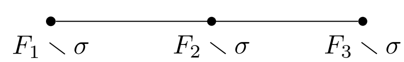


Figure 1: Nerve of Δ

So, by Lemma 2.3 the link of σ is homotopy-equivalent to a path and thus is contractible. $\square$

Statement If $F_1 \cap F_2 \cap F_3 \neq \emptyset$ and exactly three pairwise intersections are distinct and not equal to the triplewise intersection, then the link of the triplewise intersection has a hole and is therefore non-contractible. The proof for case 2c in Theorem 3.1 is as follows:

Proof. Let $F_1 \cap F_2 \cap F_3 = \sigma \neq \emptyset$. Let the pairwise intersections that are distinct from the triplewise intersection be $F_1 \cap F_2 = a, F_2 \cap F_3 = b$ and $F_1 \cap F_3 = c$. Consider the link, $\mathrm{Lk}_\sigma(\Delta)$. We define the set of facets of $\mathrm{Lk}_\sigma(\Delta)$ as $\mathcal{L}_\Delta(\sigma) = \{F_1 \setminus \sigma, F_2 \setminus \sigma, F_3 \setminus \sigma\}$. These facets have three pairwise intersections: $a^* = a \setminus \sigma, b^* = b \setminus \sigma$, $c^* = c \setminus \sigma$ and no triplewise intersection since $(F_1 \setminus \sigma) \cap (F_2 \setminus \sigma) \cap (F_3 \setminus \sigma) = (F_1 \cap F_2 \cap F_3) \setminus \sigma = \sigma \setminus \sigma = \emptyset$. Therefore, the nerve, $\mathcal{N}(\mathcal{L}_\Delta(\sigma))$, is the following simplicial complex:

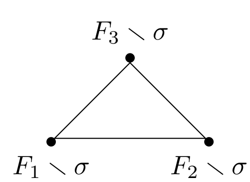


Figure 2: Nerve of Δ

$\mathcal{N}(\mathcal{L}_\Delta(\sigma))$ contains a hole and is therefore non-contractible. Thus, by Lemma 2.3, $Lk_\Delta(\sigma) \simeq \mathcal{N}(\mathcal{L}_\Delta(\sigma))$ implies $Lk_\Delta(\sigma)$ is non-contractible. $\square$

3.2 Implication for codes

In this section we discuss the implications of our contractibility results, and show that no local obstructions $\Rightarrow$ max intersection-complete and therefore no local obstructions $\Rightarrow$ convex.

Theorem 3.2. Let $\mathcal{C}$ be a neural code with three maximal codewords. Then $\mathcal{C}$ is convex $\Leftrightarrow$ $\mathcal{C}$ has no local obstructions.

Theorem 3.3. Let $\mathcal{C}$ be a neural code with exactly three maximal codewords, c_1, c_2, c_3, and assume that $\mathcal{C}$ has no local obstructions. Then,

1. If $c_1 \cap c_2 \cap c_3 = \sigma \neq \emptyset$ and there are exactly two pairwise intersections that are distinct from the triplewise intersection, then the link of σ is contractible and $\mathcal{C}$ is convex.
2. If $c_1 \cap c_2 \cap c_3 = \sigma \neq \emptyset$ and there are exactly zero, one, or three pairwise intersections that are distinct from the triplewise intersection, then the link of σ is non-contractible, thus the code is convex.
3. If $c_1 \cap c_2 \cap c_3 = \emptyset$, then the link of every intersection is non-contractible, thus the code is convex.

Proof. Let $\mathcal{C}$ be a neural code with exactly three maximal codewords, c_1, c_2, c_3 and assume there are no local obstructions.

1. In the case where $c_1 \cap c_2 \cap c_3 = \sigma \neq \emptyset$ and there are exactly two pairwise intersections that are distinct from the triplewise intersection, the link of σ is contractible and $\mathcal{C}$ is not required to be max intersection-complete in order to have no local obstructions. We show that no local obstructions implies convexity by providing, in the following section, a convex realization of $\mathcal{C}$.
2. In the case where $c_1 \cap c_2 \cap c_3 = \sigma \neq \emptyset$ and there are exactly zero, one, or three pairwise intersections that are distinct from the triplewise intersection, the link of σ is non-contractible. Since every intersection is a local obstruction if not included in $\mathcal{C}$, we must include all intersections to have no local obstructions, thus getting max intersection-complete, which implies convexity by Lemma 2.1.
3. In the case where $c_1 \cap c_2 \cap c_3 = \emptyset$, the link of every intersection is non-contractible. Similar to the previous case, since every intersection is a local obstruction if not included in $\mathcal{C}$, we must include all intersections to have no local obstructions, thus getting max intersection-complete which implies convexity by Lemma 2.1.

$\square$

3.3 Convex Realization for Codes with 2 Pairwise Intersections Distinct from the Triplewise

In this section we provide a convex realization for neural codes $\mathcal{C}$ with three maximal codewords F_a, F_b, F_c such that $F_a \cap F_b \cap F_c = \sigma \neq \emptyset$, $F_a \cap F_b \neq \sigma$, $F_b \cap F_c \neq \sigma$ and $F_a \cap F_c = \sigma$. Since the link of the triplewise intersection is contractible, as shown above, a convex realization is needed in these cases to prove convexity.

Theorem 3.4. Assume Δ is a simplicial complex with exactly three facets F_a, F_b, and F_c such that $F_a \cap F_b \cap F_c = \sigma \neq \emptyset$ with exactly two pairwise intersections distinct from the triplewise intersection. Then the construction below yields a convex realization in $\mathbb{R}$ for $\mathcal{C}_{min}(\Delta)$. ($\mathcal{C}_{min}(\Delta)$ is the smallest code with respect to inclusion with simplicial complex Δ that has no local obstructions.)

For the neural code, $\mathcal{C}$, the minimal code $\mathcal{C}_{min}(\Delta)$ consists of all facets of Δ and all codewords with non-contractible links. We have previously shown that the link of the triplewise intersection is contractible, thus the minimal code is $\mathcal{C}_{min}(\Delta) = \{\emptyset, F_a, F_b, F_c, F_a \cap F_b, F_b \cap F_c\}$. Both convex open and convex closed realizations of $\mathcal{C}_{min}(\Delta)$ can be constructed in $\mathbb{R}$ such that the codewords appear in the following order:

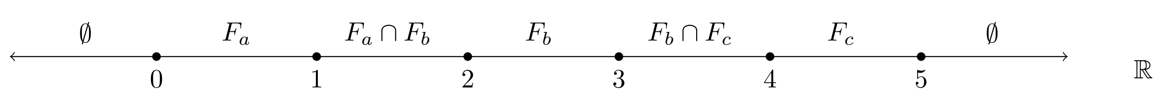


Figure 3: Realization of $\mathcal{C}_{min}(\Delta)$ in $\mathbb{R}$

The following is an explicit construction of a convex realization, $\mathcal{U} = \{\mathcal{U}_i\}$, in $\mathbb{R}$ for $\mathcal{C}_{min}(\Delta)$.

For each neuron, x, in $\mathcal{C}$:

- If $x \in F_a \cap F_b \cap F_c$, construct $\mathcal{U}_x$ as region (0,5) for open or [0,5] for closed in $\mathbb{R}$
- If $x \in F_a \cap F_b$, but $x \notin F_c$, construct $\mathcal{U}_x$ as region (0,3) for open or [0,3] for closed in $\mathbb{R}$
- If $x \in F_b \cap F_c$, but $x \notin F_a$, construct $\mathcal{U}_x$ as region (2,5) for open or [2,5] for closed in $\mathbb{R}$
- If $x \in F_a$, but $x \notin F_b$ and $x \notin F_c$, construct $\mathcal{U}_x$ as region (0,1) for open or [0,1] for closed in $\mathbb{R}$
- If $x \in F_b$, but $x \notin F_a$ and $x \notin F_c$, construct $\mathcal{U}_x$ as region (2,3) for open or [2,3] for closed in $\mathbb{R}$
- If $x \in F_c$, but $x \notin F_a$ and $x \notin F_b$, construct $\mathcal{U}_x$ as region (4,5) for open or [4,5] for closed in $\mathbb{R}$

Proof. We now show that the realization gives us the desired code, $\mathcal{C}_{min}(\Delta)$, by demonstrating that it contains all codewords in $\mathcal{C}_{min}(\Delta)$ and no other codewords. It is evident, by the way in which the realization is constructed, that each codeword appears in the following intervals of the construction:

- F_a in the interval $(0, 1)$
- F_b in the interval $(2, 3)$
- F_c in the interval $(4, 5)$
- $F_a \cap F_b$ in the interval $(1, 2)$
- $F_b \cap F_c$ in the interval $(3, 4)$
- $\emptyset$ in the interval $(5, 6)$

Now we show that no codewords exist in the realization that are not included in $\mathcal{C}_{min}(\Delta)$. We have shown that each 1-unit interval corresponds to a codeword in $\mathcal{C}_{min}(\Delta)$. Similarly, the endpoints of each interval also represent codewords in $\mathcal{C}_{min}(\Delta)$. The points left to define are the endpoints of the intervals: $1, 2, 3, 4, 5$.

In the case of a closed, convex realization, the open intervals are replaced by closed, and the points $1, 2, 3, 4, 5$ represent the codewords F_a, F_b, F_b, F_c, F_c, respectively. Each endpoint marks a location where neurons from both an intersection of maximal codewords and one of the maximal codewords meet. The neurons in the intersections of two maximal codewords must be contained in the maximal codewords themselves, therefore each endpoint holds only the neurons in the larger facet. Thus, the endpoint represents the maximal codeword.

In the case of a convex open realization, the points $1, 2, 3, 4, 5$ represent the codewords $F_a \cap F_b$, $F_a \cap F_b$, $F_b \cap F_c$, $F_b \cap F_c$, and $\emptyset$ respectively. The neurons that exist at each endpoint are only those neurons whose interval extends to either side of the endpoint. An examination of the intervals defined above provides the neurons at each endpoint to be the neighboring intersection of maximal codewords and for the case of 5, the empty set. $\square$

Example 3.1. Consider the neural code, $\mathcal{C}_{min}(\Delta) = \{\emptyset, 123, 124, 1356, 13, 12\}$. Following Theorem 3.3, $\mathcal{C}_{min}(\Delta)$ can be realized in $\mathbb{R}$ in the following way:

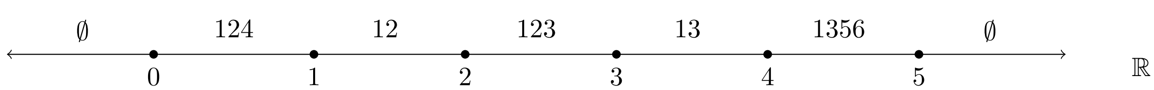


Figure 4: Realization of the code in $\mathbb{R}$ (codewords are labeled)

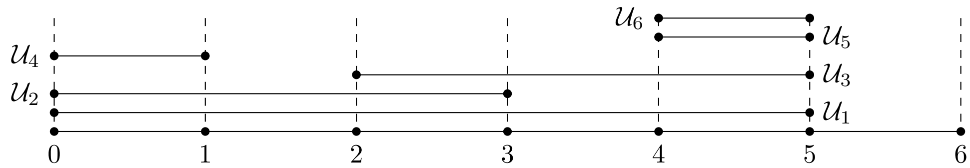


Figure 5: Arrangement of individual neurons of code in $\mathbb{R}$

The example includes all of the necessary codewords, with no extra codewords appearing. It can be considered convex open or convex closed.

3.4 Convex Construction in $\mathbb{R}^2$

In the previous section, we provided both convex open and convex closed realizations in $\mathbb{R}$ for minimal codes with three maximal codewords, a non-empty triplewise intersection, and exactly two unique pairwise intersections distinct from the triplewise intersection. We will show how this realization can be used to construct convex open and convex closed realizations for any code $\mathcal{C}$ such that $\mathcal{C}_{min}(\Delta) \subseteq \mathcal{C} \subseteq \Delta$. We start with the case where there are two distinct pairwise intersections.

Theorem 3.5. Let Δ be a simplicial complex with three facets, F_a, F_b, F_c such that the triplewise intersection is nonempty and there are exactly two nonempty, pairwise intersections distinct from the triplewise intersection. Then, for any code $\mathcal{C}$ such that $\mathcal{C}_{min}(\Delta) \subseteq \mathcal{C} \subseteq \Delta$, there is a convex open and convex closed realization of $\mathcal{C}$ in $\mathbb{R}^2$.

Proof. We will provide an explicit convex open and convex closed construction for a code $\mathcal{C}$ and show that the construction contains all codewords in $\mathcal{C}$ and no additional codewords.

Start with the realization of the minimal code, $\mathcal{C}_{min}(\Delta) = \{F_a, F_b, F_c, F_a \cap F_b, F_b \cap F_c\}$, in $\mathbb{R}$ which can be achieved following the construction in Theorem 3.4. Expand the 1-dimensional realization to a ball in $\mathbb{R}^2$, as shown below in Figure 6.

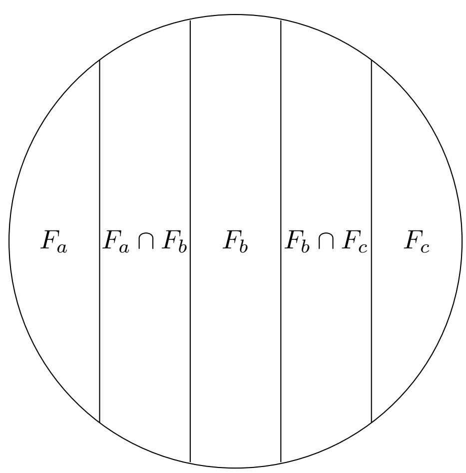


Figure 6: Realization of $\mathcal{C}_{min}(\Delta)$ in $\mathbb{R}^2$

It is clear by Theorem 3.4 that the construction contains all codewords $c_i \in \mathcal{C}_{min}(\Delta)$ and no additional codewords. Now, all codewords $c_j \in \mathcal{C} \setminus \mathcal{C}_{min}$ can be added to the realization by using lines to "slice" the boundary of regions of the ball and "roll back" the receptive fields of certain neuron(s) to obtain a region for the desired codeword.

More specifically, let S be the set of all codewords $c_j \in \mathcal{C} \setminus \mathcal{C}_{min}(\Delta)$. By definition, all $c_j \in S$ are contained in some facet, $F_i \in \mathcal{C}_{min}$. Thus, our set S can be partitioned into three subsets: S_1, S_2, and S_3 where the elements of the subset S_i are all codewords $c_j \in S$ such that $c_j \subseteq F_i$. Let n_i be the number of elements of subset S_i. We will proceed by induction on n_i to show that, for an arbitrary set S_i, all codewords are realized in the construction.

The base case of $n_i = 0$ is simple in its construction. There are no codewords contained in S_i and therefore, no additional regions to be constructed in the realization.

Consider the $n_i = 1$ case where there is one codeword $c_1 \in S_i$. Construct a region for this codeword in the realization by selecting two points a_1, b_1 on the circle such that $a_1, b_1 \in U_{F_i}$. Let a_1 be the point adjacent to the endpoint of the arc of the circle containing all points in U_{F_i}. Let b_1 be the mid point of this arc. Let L be a line through a_1 and b_1 such that $L \cap U_\sigma \neq \emptyset$ if and only if $\sigma = F_i$. This line breaks $\mathbb{R}^2$ into a convex, closed half-space that only intersects the ball in the region corresponding to c_1 and a convex, open half-space, H, that intersects all other regions of the ball. For all neurons $j \in F_i \setminus c_1$, replace the receptive field U_j by $U_j \cap H$. This leaves a region with only neurons contained in c_1 as desired.

We now assume that we can form such regions for $n_i = k$ codewords and show that the construction holds for the case of $n_i = k+1$.

By the inductive hypothesis, there is a realization for $n_i = k$ codewords. We follow the same steps described above to add the additional codeword c_{k+1}. Consider the arc of the circle containing all points in U_{F_i}. Select two points, $a_{k+1}, b_{k+1} \in U_{F_i}$, along this arc such that a_{k+1} is the point adjacent to the point b_k and b_{k+1} is the midpoint of the arc containing all points in U_{F_i}. Let L_{k+1} be the line segment connecting a_{k+1} and b_{k+1}. Again, we consider the half-spaces formed by the line in $\mathbb{R}^2$. The closed half-space that intersects with the ball only in the region corresponding to c_{k+1}. The open half-space, H, intersections all other regions of the ball. For all neurons $j \in F_i \setminus c_{k+1}$, replace the receptive field U_j be $U_j \cap H$. This results in a region corresponding to c_{k+1} as desired.

Thus, by induction, we can realize any number of codewords n_i in each C_i'.

An example of lines "slicing" the realization of the minimal code realization in $\mathbb{R}^2$ to create new codewords c_1, c_2 is shown in Figure 7 below:

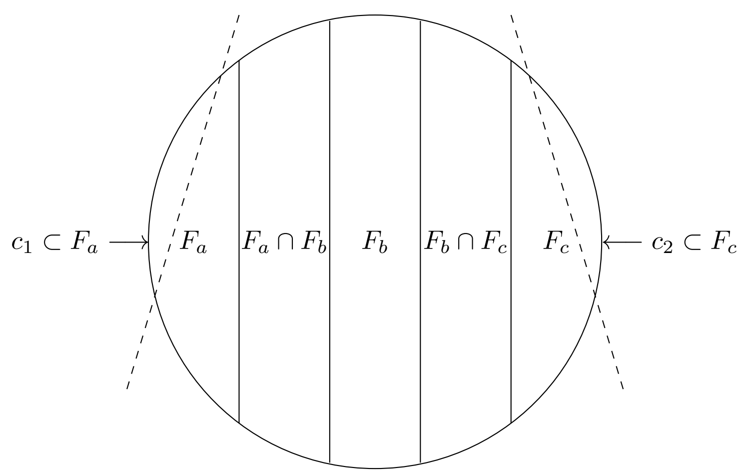


Figure 7: Realization of the code $\mathcal{C} = \{F_a, F_b, F_c, F_a \cap F_b, F_b \cap F_c, c_1, c_2\}$

To complete the proof, we show that there are no regions corresponding to codewords not contained in $\mathcal{C}$.

The only areas where new codewords may appear is on the boundary between two existing regions. The proof of Theorem 3.4 shows that no additional codewords appear on the boundaries between the regions corresponding to the facets and their intersections. We now consider the boundaries created when additional codewords are added.

By the construction, for any additional codeword c, the region corresponding to c shares a single boundary with the region corresponding to the facet, F_i, of which c is a subset.

- For a convex open realization, the boundary corresponds to the neurons that are shared between c and F_i, which is all neurons in c and therefore the boundary represents the codeword c which is in $\mathcal{C}$.

- For a convex closed realization, the boundary corresponds to all neurons in c and all neurons in F_i. Since all neurons in c are contained in F_i, the boundary represents exactly the neurons in F_i and therefore the codeword F_i which is in $\mathcal{C}$.

Thus, the realization in $\mathbb{R}^2$ given be the above construction, contains every codeword in $\mathcal{C}$ and no additional codewords. We have then provided a construction of an open convex and closed convex realization of $\mathcal{C}$ in $\mathbb{R}^2$. ☐

Example 3.2. Consider the code from Example 3.1, $\mathcal{C}_{min}(\Delta) = \{\emptyset, 123, 124, 1356, 13, 12\}$. By Theorem 3.4, the code, $\mathcal{C}$, obtained by adding the codewords $\{24, 23, 6, 5\}$ can be realized in $\mathbb{R}^2$ as follows:

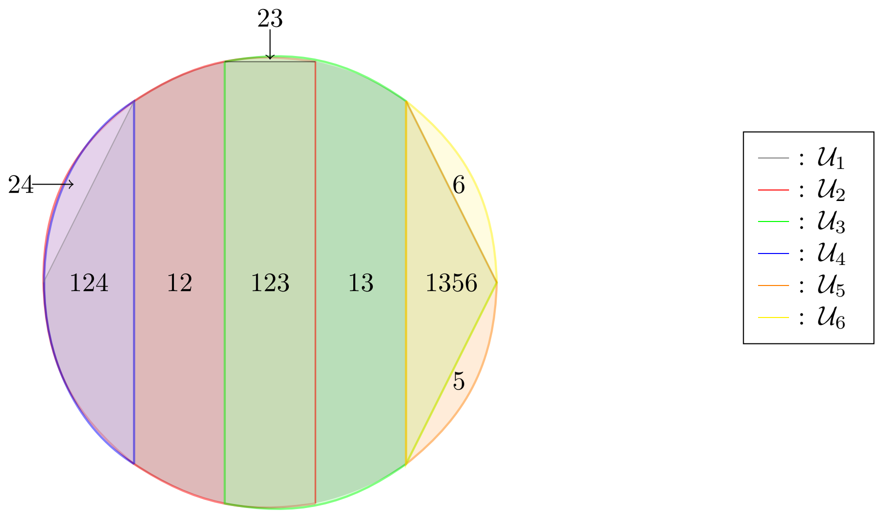


Figure 8: Realization of $\mathcal{C} = \{\emptyset, 124, 123, 1356, 12, 13, 5, 6, 23, 24\}$ in $\mathbb{R}^2$

3.5 Embedding Dimension of Minimal Codes

Below we generalize the minimal embedding dimension for codes with 3 maximal codewords. By Lemma 2.3, such codes can be realized in $\mathbb{R}^2$, however we make further specifications indicating certain minimal codes can be realized in $\mathbb{R}^1$.

Theorem 3.6. Let Δ be a simplicial complex with exactly 3 facets F_a, F_b, and F_c.

1. If $F_a \cap F_b \cap F_c = \emptyset$ or $F_a \cap F_b \cap F_c = \sigma \neq \emptyset$ and each pairwise intersection, $F_a \cap F_b$, $F_a \cap F_c$, and $F_b \cap F_c$, is distinct from $F_a \cap F_b \cap F_c$, then the minimal embedding dimension is 2.

2. In all other cases, the minimal embedding dimension is 1.

Proof. Let Δ be a simplicial complex with facets F_a, F_b, and F_c, and let $C_{min}(\Delta)$ be the neural code of simplicial complex Δ.

1. If there are exactly 3 pairwise intersections that are distinct from the triplewise intersection, then the minimal code has both convex open and convex closed realizations in $\mathbb{R}^2$, as shown by the image below.

   Let $\mathcal{C}_{min}$ be a neural code with maximal codewords F_a, F_b, and F_c such that $F_a \cap F_b \cap F_c = \emptyset$, $F_a \cap F_b = \alpha$, $F_a \cap F_c = \beta$, and $F_b \cap F_c = \gamma$. Then the following minimal code has both convex open and convex closed realizations in $\mathbb{R}^2$ as shown below. For cases in which $F_a \cap F_b \cap F_c = \sigma$ and all other criteria hold, the middle region changes from $\emptyset$ to σ.

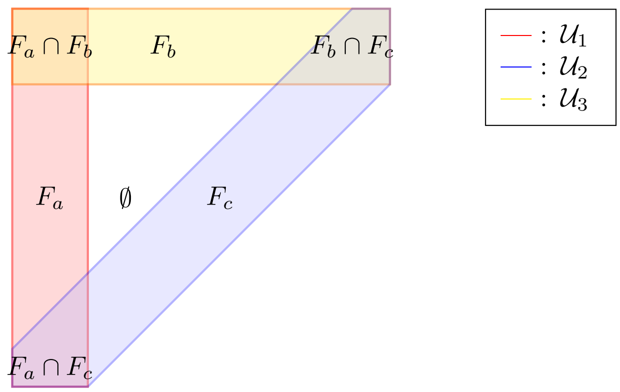


Figure 9: Realization of $\mathcal{C}_{min}$ in $\mathbb{R}^2$

2. If there are 0, 1, or 2 pairwise intersections that are distinct from the triplewise intersection, then the minimal code has both convex open and convex closed realizations in $\mathbb{R}$.

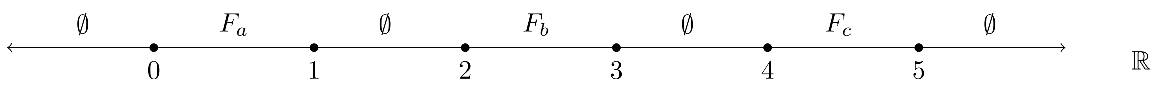


Figure 10: Realization of $\mathcal{C}$ in $\mathbb{R}$ ($F_a \cap F_b \cap F_c = \emptyset$, 0 pairwise intersections)

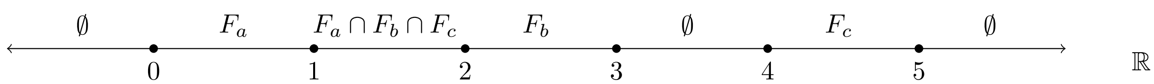


Figure 11: Realization of $\mathcal{C}$ in $\mathbb{R}$ ($F_a \cap F_b \cap F_c = \sigma$, 0 pairwise intersections)

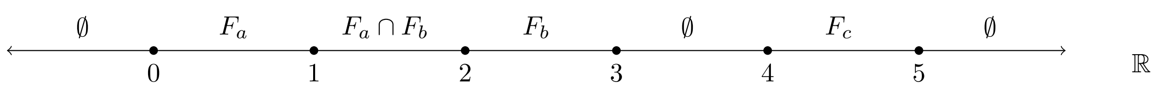


Figure 12: Realization of $\mathcal{C}$ in $\mathbb{R}$ ($F_a \cap F_b \cap F_c = \emptyset$, 1 pairwise intersection)

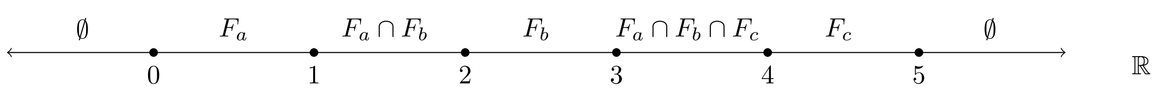


Figure 13: Realization of $\mathcal{C}$ in $\mathbb{R}$ ($F_a \cap F_b \cap F_c = \sigma$, 1 pairwise intersections)

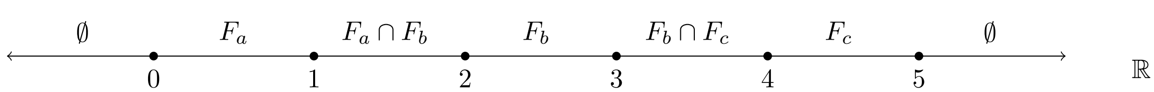


Figure 14: Realization of $\mathcal{C}$ in $\mathbb{R}$ ($F_a \cap F_b \cap F_c = \emptyset$, 2 pairwise intersections)

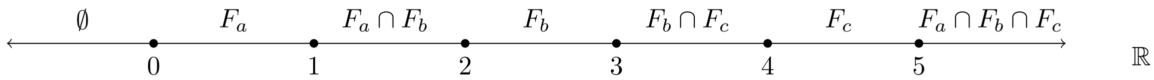


Figure 15: Realization of $\mathcal{C}$ in $\mathbb{R}$ ($F_a \cap F_b \cap F_c = \sigma$, 2 pairwise intersections)

In cases where the embedding dimension is 1, *Convex Neural Codes in* $\mathbb{R}$ [5] offers further characterization of the codes.

$\square$

Below is a table that summarizes the minimal embedding dimensions for codes with 3 maximal codewords:

Table 1: Minimal embedding dimension of $\mathcal{C}_{min}(\Delta)$, based on the number of pairwise intersections that are distinct from the triplewise intersection

Embedding Dimension	**Number of Pairwise Intersections Distinct from the Triplewise**			
	0	**1**	**2**	**3**
1	✓	✓	✓	
2				✓

3.6 Minimal Embedding Dimension

In this section, we expand upon the results of the previous table. That table indicates the smallest embedding dimension for minimal codes with 3 facets. An expansion similar to that shown in Section 3.4, has both a convex open and convex closed realization for any code, $\mathcal{C}$, with $\mathcal{C}_{min}(\Delta) \subseteq \mathcal{C} \subseteq \Delta$.

Theorem 3.7. If $\mathcal{C}$ is a neural code with exactly three maximal codewords, then the minimal embedding dimension is at most 2.

Proof. By Theorem 3.6, if a code $\mathcal{C}$ has exactly three maximal codewords, then the embedding dimension of $\mathcal{C}_{min}(\Delta)$ is 1 or 2.

If $\mathcal{C}_{min}(\Delta)$ has an embedding dimension of 1. Then, in the proof of Theorem 3.4, we provided an explicit construction in $\mathbb{R}^2$ of the code $\mathcal{C}$ whose minimal code $\mathcal{C}_{min}(\Delta)$ can be realized in $\mathbb{R}$. Otherwise, if the embedding dimension of $\mathcal{C}_{min}(\Delta)$ is 2, the code $\mathcal{C}$ can be realized in $\mathbb{R}^2$ following a similar construction to that provided in the proof of Theorem 3.4. First start with the realization of the minimal code in $\mathbb{R}^2$ as depicted in Figure 9. The rectangular regions can be morphed into elliptical regions as shown below in Figure 16. From here, the construction and proof is the same as in Theorem 3.4. Lines are used to intersect regions of the ball to "rollback" neurons and create regions for the new codewords.

Thus, in both cases, the embedding dimension of the full code $\mathcal{C}$ is $\mathbb{R}^2$ as desired. $\square$

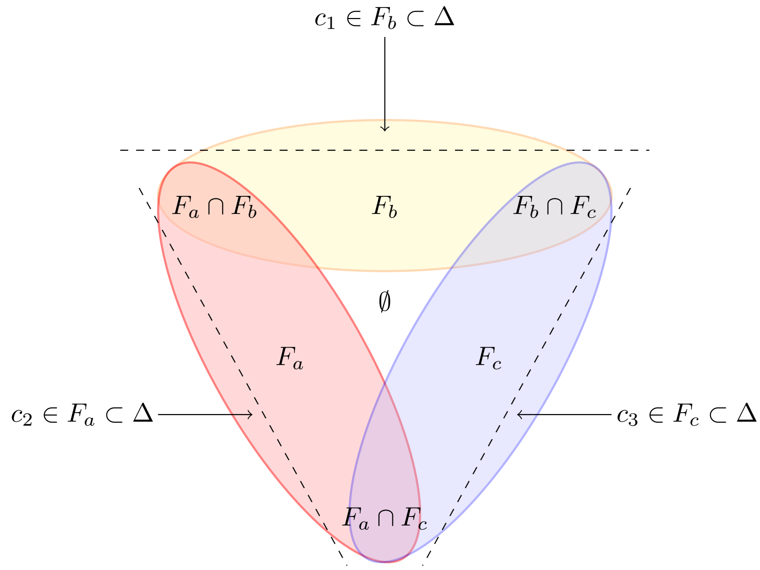


Figure 16: Realization of C in $\mathbb{R}^2$ when there are 3 pairwise intersections

Figure 16 depicts the convex realization of the mode $\mathcal{C} = \{\emptyset, F_a, F_b, F_c, F_a \cap F_b, F_a \cap F_c, F_b \cap F_c, c_1, c_2, c_3\}$ in $\mathbb{R}^2$ constructed using the construction provided in the proof of Theorem 3.7.

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