Solving Trinomials over $\mathbb{Q}_p$

Elliott Fairchild

July 27, 2021

Solving Trinomials

Problem

Let $f(x) = c_1x^{a_1} + c_2x^{a_2} + c_3x^{a_3} \in \mathbb{Z}[x]$*. How many roots of* f *over* $\mathbb{Z}/(p^k)$ *are there, and where do they lie?*

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Example

Let $f(x) = x^2$. Then f has a single degenerate root at 0 over $\mathbb{Z}/(p)$, but over $\mathbb{Z}/(p^2)$, the roots are given by $(0, p, \ldots, (p-1)p)$.

Applications to Coding Theory

- Just as strings of bits can represent words and data, we can consider a more general code K written as a tuple $(q_1, \ldots, q_\rho)$ of elements of $\mathbb{Z}/(p^k)$.

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- Applications in error-correction involve computing roots of a polynomial $G \in (\mathbb{Z}/(p^k))[x][y]$ over $(\mathbb{Z}/(p^k))[x]$.

Passing to $\mathbb{Q}_p$

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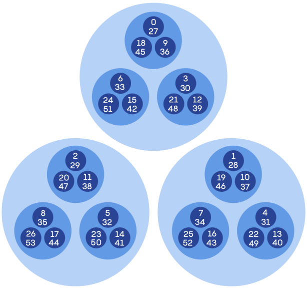

Figure 1: 3-adic integers (Quanta Magazine, 2020)

- Observe we can uniquely write any rational $\frac{a}{b}$ as $\frac{a}{b} = p^k \frac{n}{d}$, where $k \in \mathbb{Z}$ and $\gcd(n, d) = 1$. The p-adic valuation $\text{ord}_p(\cdot)$ is defined on $\mathbb{Q}$ to be $\text{ord}_p(a/b) = k$.

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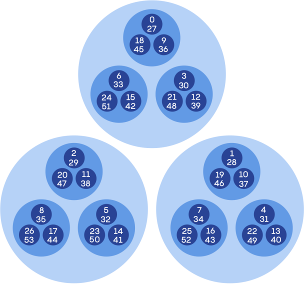

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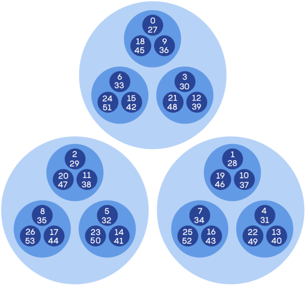

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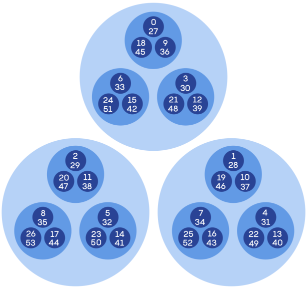

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- The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$, the p-adic numbers.
- p-adic numbers can also be expressed by formal series $\sum_{j=s}^{\infty} a_j p^j$, where $a_j \in \{0, \ldots, p-1\}$

An Analogy

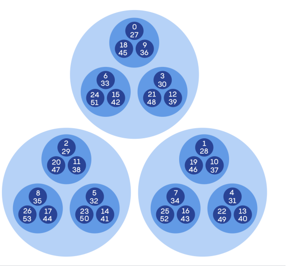

Figure 2: 3-adic integers (Quanta Magazine, 2020)

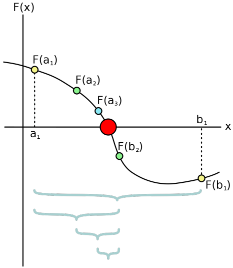

Figure 3: Bisection Method (Wikipedia, 2021)

- Consider the sequence obtained by extracting the digits of the non-1 root of $x^2 - 1$ over $\mathbb{Z}_3$: 2, $2 + 2 \cdot 3$, $2 + 2 \cdot 3 + 2 \cdot 3^2$, ...
- Consider the sequence obtained by applying the bisection method to $\sqrt{2}$ in the interval $[1, 2]$: 1, 1.25, 1.375, 1.4375, ...
- Both sequences converge at a geometric rate! Applying Newton's method to either allows both to converge even faster!

Definition

Let $f \in \mathbb{Z}[x]$ and let $\tilde{f}$ be its reduction mod p.

An example over $\mathbb{Q}_{17}$:

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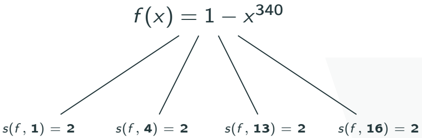

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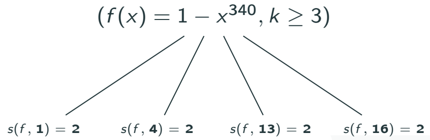

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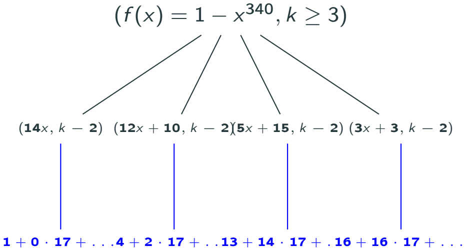

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An example over $\mathbb{Q}_3$:

$(f_0(x) = x^9 - 1, k_0 \geq 3)$

How to solve over $\mathbb{Q}_p$: Trees

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$(f_1, k - 2)$, $\tilde{f}_1 = x$

Trees and Binomials

Theorem (Rojas and Zhu, 2021)

Following the notation of $\mathcal{T}_{p,k}(f)$ above, let $f = f_{0,0} = c_0 + c_1 x^d \in \mathbb{Z}[x]$ with $c_0 c_1 \not\equiv 0 \mod p$. Then for all k, the tree $\mathcal{T}_{p,k}(f)$ has depth at most 1.

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- The tree gives approximate roots of f in just two digits!
- This gives complexity of root-approximating algorithms linear in $gcd(d, p-1)$ and polynomial in $\log(dpH)$, where $H = \max\{c_0, c_1\}$
- Also, the roots are never less than $1/p$ apart.

Theorem (Rojas and Zhu, 2021)

Let $f = c_1 + c_2 x^{a_2} + c_3 x^{a_3}$ *be a trinomial with* $0 < a_2 < a_3$, $p \nmid c_1$. *Define* $S_0 = \max\{s(f, \zeta_0) \mid \zeta_0$ *is a degenerate root of* f *over* $\{0, 1, \ldots p-1\}\}$ *and* $D = \max\{ord_p(\zeta - \xi) \mid \zeta, \xi$ *are non-degenerate roots of* f *over* $\mathbb{Q}_p\}$, *setting either quantity to* 0 *if not applicable. Then* $k \geq 1 + S_0 \min\{1, D\} + M_p \max\{D-1, 0\}$ *(where* $M_p = 4$, 3, *or* 2, *according to* $p = 2$, $p = 3$, $p \geq 5$*) guarantees* $\mathcal{T}_{p,k}$ *has depth at least* D.

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- Explicit, but worse (not $O(1)$) on k than in the binomial case.
- The analogous root spacing bound induced is given by $|\log|z_1 - z_2|_p| = O(p\log^2(dH)\log_p(d))$.
- Two simple families of examples prove that the minimal root spacing is at least linear in $\log(dH)$ and that the depth of k has dependence on D and S_0 .

Two families of examples

Example

The family $g_p(x) = x^2 - (2 + p^j)x + (1 + p^j)$ has roots $z_1 = 1$, $z_2 = 1 + p^j$, so that $\log|z_1 - z_2|_p = -\log(H - 2)$.

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Example

Similarly, we can prove family $h_p(x) = x^{p^j+2} - 2x + 1$ has roots $z_1 = 1$, $z_2 = 1 + (p - 1)p^j + \ldots$ (so that $\log |z_1 - z_2|_p = -\log(d - 2)$) and extremal k.

Acknowledgements

* Professor Rojas
* TAs and Professors
* TAMU and NSF

Thank you for listening!