Discriminant Varieties of Arbitrary Degree Univariate Tetranomials

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Project Goal Introduction

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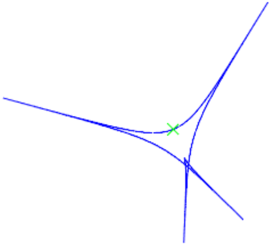

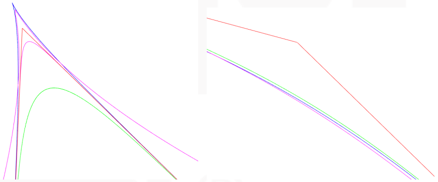

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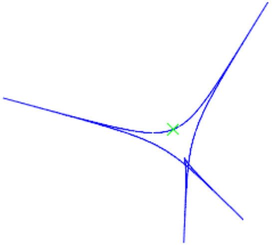

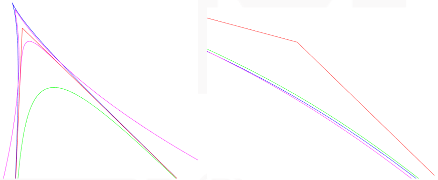

...but what does that mean? Let's start with some definitions!

Project Background

We are working with univariate tetranomials. We care more about the number of variables and terms than about degree.

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Definition

We call

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an n-variate $n+k$-nomial, with f $\in \mathbb{C}[\mathrm{x}_1 \ldots x_n]$ and $c_i \neq 0$.The set A $= \{a_1 \ldots . a_{n+k}\} \subset \mathbb{Z}$ is the support of the polynomial.

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We will use something called the discriminant to understand the positive zero set of our polynomials...

Project Background continued

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Given an n-variate $n+k$-nomial, with support A, the A-discriminant variety is the closure of $\nabla_A = (c_1,...,c_{n+k}) \in (\mathbb{C}^*)^{n+k}$, where $f = c_1 x^{a_1} ... c_{n+k} x^{a_{n+k}}$ has a *degenerate root*.

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1. Discriminant polynomial
2. Issues with computing
3. Efficient solution?

Project Background continued

We parameterize the discriminant variety using the Horn-Kapranov Uniformization:

1. Support matrix A
2. Form matrix B from basis of right nullspace

Theorem

The image of $\Psi_{A,B}([\lambda]) = log|\lambda B^T|B$, *with* $[\lambda] \in \mathbb{P}_{\mathbb{C}}^{k-2}$, *is a slice of* $log|\nabla_A|$

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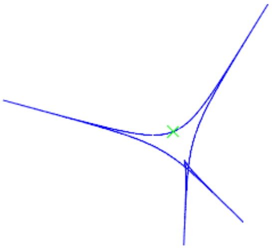
Example: $A = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

The parameterization we get is $(log|\lambda_1 + 3\lambda_2| - 2log|2\lambda_1 + 3\lambda_2| +$ log$|\lambda_1|, 2log|\lambda_1 + 2\lambda_2| - 3log|2\lambda_1 + 3\lambda_2| + log|\lambda_2|)$...

Project Background continued

...that parameterization is what produced the plot from the first slide!



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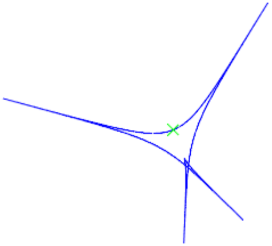

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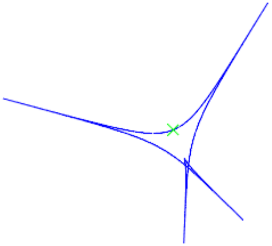

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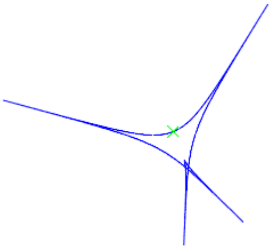

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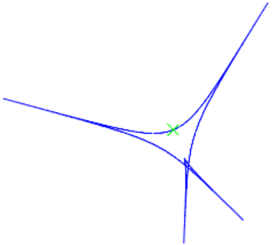

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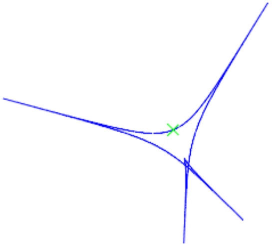

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5. Approximations of the reduced A-discriminant variety

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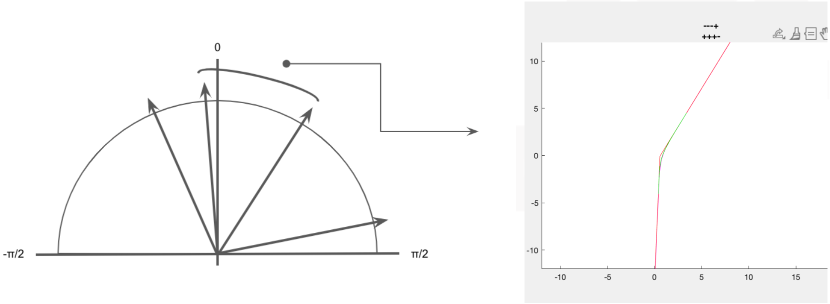

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3. Now, we apply rotations given by the rays in each orthant...

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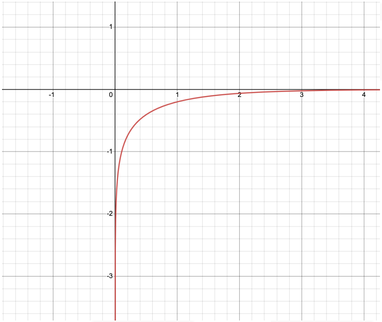

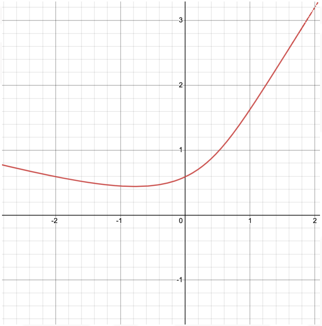

After we have applied the proper rotations given by the rays, we compute the constant determining the sharpness of the curve from the angle formed by the rays.

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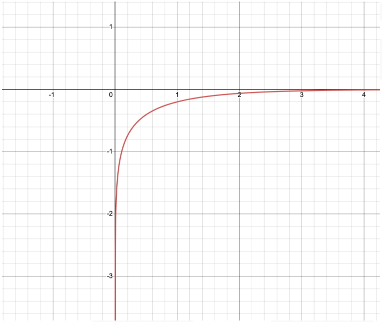
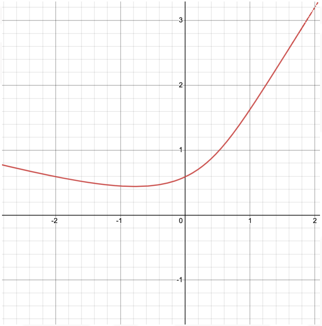


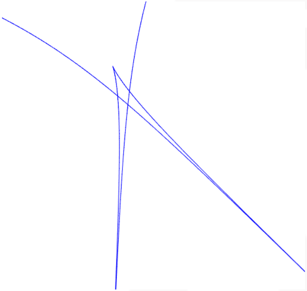


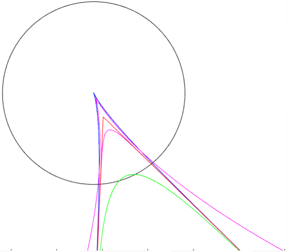
After we have applied the proper rotations given by the rays, we compute the constant determining the sharpness of the curve from the angle formed by the rays.

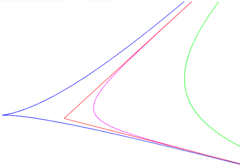
In most orthants, this curve matches nearly perfectly with the one parameterized by HKU...but what about cusps?

Implementing Our New Approximation

We call critical points of the map given by HKU *cusps*.







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So, we use it only when the inputted point is outside of the circle shown above. When inside, we use another familiar curve, of the form $y^3 = x^2$

Once again, we apply rotation and sharpen the curve according to the rays and the angle they form. The shape is not always symmetrical, so a little trick is needed there.

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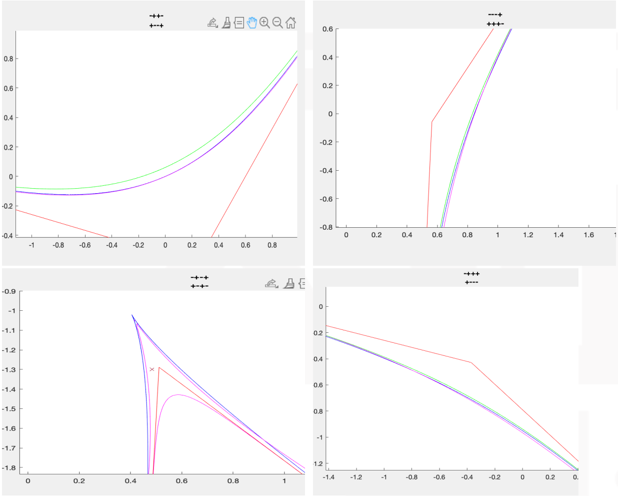
So, in each orthant we have approximated the reduced discriminant variety. Now, determining sidedness is simple:

1. Take $log|\bullet|$ of input point in 4D coefficient space
2. Multiply by B matrix
3. Identify proper orthant
4. Evaluate expression approximating curve in that orthant
5. Number of zeroes is given by Viro diagram

Implementing Our New Approximation



Example: input point [-0.05, 0.8, -3, 3], produces output 3 real, positive roots

References

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Thank you for listening!