

FALTINGS HEIGHTS OF CM ELLIPTIC CURVES AND SPECIAL GAMMA VALUES

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ABSTRACT. In this paper, we evaluate the Faltings height of an elliptic curve with complex multiplication by an order in an imaginary quadratic field in terms of values of Euler's Gamma function at rational numbers.

1. INTRODUCTION

Let L be a number field with ring of integers \mathcal{O}_L . Let E/L be an elliptic curve over L and let $\mathcal{E}/\mathcal{O}_L$ be a Néron model for E/L . Then the *Faltings height* of E/L is defined by

$$h_{\text{Fal}}(E/L) := \frac{1}{[L : \mathbb{Q}]} \deg(\omega_{\mathcal{E}/\mathcal{O}_L}),$$

where $\omega_{\mathcal{E}/\mathcal{O}_L} = s^* \Omega_{\mathcal{E}/\mathcal{O}_L}$ is the metrized line bundle on $\text{Spec}(\mathcal{O}_L)$ given by the pullback of the sheaf of Néron differentials $\Omega_{\mathcal{E}/\mathcal{O}_L}$ by the zero section $s : \text{Spec}(\mathcal{O}_L) \rightarrow \mathcal{E}$ (see Section 6). Here the Faltings height is normalized as in [Sil86].

Now, if E/L has complex multiplication by the ring of integers \mathcal{O}_K of an imaginary quadratic field K , Deligne [Del85] evaluated the *stable* Faltings height of E/L in terms of values of Euler's Gamma function $\Gamma(s)$ at rational numbers (see the discussion in Remark 1.2). He used this result to calculate the minimum value attained by the stable Faltings height.

In this paper, we will give a similar formula for the Faltings height of an elliptic curve E/L with complex multiplication by *any* order in K (not necessarily maximal). To state this result, let K be an imaginary quadratic field of discriminant D with ideal class group $\text{Cl}(D)$, unit group \mathcal{O}_K^\times , and Kronecker symbol χ_D . Let $h(D) = \#\text{Cl}(D)$ be the class number and $w_D = \#\mathcal{O}_K^\times$ be the number of units. Suppose that E/L has complex multiplication by an order $\mathcal{O}_f \subset K$ of conductor $f \in \mathbb{Z}^+$ and discriminant $\Delta_f = f^2 D$. Let $\Delta_{E/L}$ be the minimal discriminant of E/L (which is an integral ideal of L) and let $j(E)$ be the j -invariant of E/L . Assume that the coefficients of the Weierstrass equation for E/L lie in the subfield $\mathbb{Q}(j(E)) \subset L$.

Theorem 1.1. *With notation and assumptions as above, we have*

$$h_{\text{Fal}}(E/L) = \log \left(N_{L/\mathbb{Q}}(\Delta_{E/L})^{1/12[L:\mathbb{Q}]} \left(\frac{\sqrt{|\Delta_f|}}{\pi} \right)^{1/2 \frac{|D|}{f}} \prod_{k=1}^{|D|} \Gamma \left(\frac{k}{|D|} \right)^{-\chi_D(k) w_D / 4h(D)} \prod_{p|f} p^{e(p)/2} \right),$$

where

$$e(p) := \frac{(1 - p^{\text{ord}_p(f)})(1 - \chi_D(p))}{p^{\text{ord}_p(f)-1}(1-p)(\chi_D(p) - p)}.$$

Remark 1.2. The Faltings height of E/L depends on the number field L . This dependence can be eliminated by passing to a finite extension L'/L such that E/L' has everywhere semistable reduction. In particular, one defines the stable Faltings height by

$$h_{\text{Fal}}^{\text{stab}}(E/L) := h_{\text{Fal}}(E/L').$$

The stable Faltings height is independent of both the number field L and the choice of extension L'/L .

Now, assume that E/L is a CM elliptic curve with everywhere good reduction, satisfying the hypotheses of Theorem 1.1. Moreover, assume that E has complex multiplication by the maximal order \mathcal{O}_K in K . Then the everywhere good reduction assumption implies that $h_{\text{Fal}}^{\text{stab}}(E/L) = h_{\text{Fal}}(E/L)$ and that $\Delta_{E/L} = \mathcal{O}_L$. Also, since \mathcal{O}_K has conductor $f = 1$, then Theorem 1.1 gives

$$h_{\text{Fal}}^{\text{stab}}(E/L) = \log \left(\left(\frac{\sqrt{|D|}}{\pi} \right)^{1/2} \prod_{k=1}^{|D|} \Gamma \left(\frac{k}{|D|} \right)^{-\chi_D(k)w_D/4h(D)} \right). \quad (1.1)$$

On the other hand, Deligne [Del85, p. 27] defined a different normalization of the stable Faltings height, which he called the *geometric height* of E and denoted by $h_{\text{geom}}(E)$. It can be shown that

$$h_{\text{geom}}(E) = h_{\text{Fal}}^{\text{stab}}(E/L) + \frac{1}{2} \log \pi. \quad (1.2)$$

Deligne then observed (see [Del85, p. 29]) that the classical Chowla-Selberg formula [CS67] can be used to prove that

$$\exp(h_{\text{geom}}(E))^{-2} = \frac{1}{\sqrt{|D|}} \prod_{k=1}^{|D|} \Gamma \left(\frac{k}{|D|} \right)^{\chi_D(k)w_D/2h(D)}. \quad (1.3)$$

If we substitute the evaluation of $h_{\text{Fal}}^{\text{stab}}(E/L)$ from (1.1) into equation (1.2) and then exponentiate, we recover Deligne's result (1.3) as a special case of Theorem 1.1 when the elliptic curve has complex multiplication by the maximal order \mathcal{O}_K .

An important step in the proof of Theorem 1.1 is a Chowla-Selberg formula for any order in K (not necessarily maximal). An arithmetic-geometric proof of such a formula was given by Nakajima and Taguchi [NT91] by employing a theorem of Faltings which relates the Faltings heights of two isogenous abelian varieties. Kaneko briefly outlined an analytic approach to the same formula in the research announcement [Kan90]. Here we give a detailed analytic proof of a Chowla-Selberg formula for orders in K . This proof is based on a renormalized Kronecker limit formula for the non-holomorphic $SL_2(\mathbb{Z})$ Eisenstein series, a period formula which relates the zeta function of an order in K to values of the Eisenstein series at CM points corresponding to classes in the ideal class group of the order, and a factorization of the zeta function of an order given by Zagier [Zag77], and in an equivalent but different form by Kaneko [Kan90].

2. AN EXAMPLE OF THEOREM 1.1

In this section, we use Theorem 1.1 and SageMath [S⁺09] to explicitly calculate both the unstable and the stable Faltings height of a CM elliptic curve defined over $L = \mathbb{Q}(\sqrt{6})$.

Let $K = \mathbb{Q}(\sqrt{-2})$ be the imaginary quadratic field of discriminant $D = -8$. Let $\mathcal{O}_K = \mathbb{Z}[\sqrt{-2}]$ be the ring of integers and let

$$\mathcal{O}_3 = \mathbb{Z} + 3\mathcal{O}_K = \mathbb{Z}[3\sqrt{-2}]$$

be the order of conductor $f = 3$ in K . Kida [Kid01] computed tables of elliptic curves with everywhere good reduction over quadratic fields. In particular, the first entry in [Kid01, Table 4, p. 557] with the choices $m = 6$,

$$j = j(3\sqrt{-2}) = 188837384000 + 77092288000\sqrt{6},$$

and

$$u = 9600500 + 3894730\sqrt{6}$$

gives the elliptic curve

$$\begin{aligned} E/L : y^2 = x^3 + (-213956647452306361344000 - 87347435555321131008000\sqrt{6})x \\ - 53870596781293500420067393011712000 \\ - 21992579042231152861893869174784000\sqrt{6}, \end{aligned}$$

which is defined over L , has j -invariant $j(E) = j$, and complex multiplication by the non-maximal order \mathcal{O}_3 . The minimal discriminant ideal of E/L is

$$\begin{aligned} \Delta_{E/L} = (-2504693124549627605027212944424000000 \\ + 804798587063718019634525385680000000\sqrt{6})\mathcal{O}_L. \end{aligned}$$

The norm of $\Delta_{E/L}$ is given by

$$N_{L/\mathbb{Q}}(\Delta_{E/L}) = 2^{18} \cdot 5^{12} \cdot 7^{12} \cdot 23^6 \cdot 29^6 \cdot 47^6 \cdot 53^6 \cdot 71^6.$$

On the other hand, letting $L' = L(\sqrt{u}) = \mathbb{Q}(\sqrt{u})$, the same entry in Kida's table gives the quartic twist

$$\begin{aligned} E^u/L' : y^2 = x^3 + (-78385716591330516253647775671705600000 \\ - 32000834795195545336704672717619200000\sqrt{6})x \\ - 377762209467237733195528032891831932481324956254208000000 \\ - 154220776216851533553188975993060485857126526746624000000\sqrt{6}, \end{aligned}$$

which is defined over L' and isomorphic to E/L over L' , has j -invariant $j(E^u) = j(E) = j$, and complex multiplication by the order \mathcal{O}_3 . Moreover, E^u/L' has minimal discriminant ideal $\Delta_{E^u/L'} = \mathcal{O}_{L'}$, and thus E^u/L' has everywhere good reduction over L' .

Now, since the discriminant of K is $D = -8$ and the conductor of the order \mathcal{O}_3 is $f = 3$, we see that $\Delta_3 = -72$, $w_{-8} = 2$ and $h(-8) = 1$. The Kronecker symbol values are $\chi_{-8}(k) = 1$ for $k = 1, 3$ and $\chi_{-8}(k) = -1$ for $k = 5, 7$, and hence $e(3) = 0$. Therefore, noting that the coefficients of E/L are contained in $\mathbb{Q}(j(E)) = L$, Theorem 1.1 gives us

$$h_{\text{Fal}}(E/L) = \log \left(N_{L/\mathbb{Q}}(\Delta_{E/L})^{1/12[L:\mathbb{Q}]} \left(\frac{\sqrt{|\Delta_3|}}{\pi} \right)^{1/2} \prod_{k=1}^8 \Gamma \left(\frac{k}{8} \right)^{-\chi_{-8}(k)/2} \right).$$

After expanding, we get

$$h_{\text{Fal}}(E/L) = \log \left(2^{3/4} 5^{1/2} 7^{1/2} 23^{1/4} 29^{1/4} 47^{1/4} 53^{1/4} 71^{1/4} \left(\frac{6\sqrt{2}}{\pi} \right)^{1/2} \left(\frac{\Gamma(5/8)\Gamma(7/8)}{\Gamma(1/8)\Gamma(3/8)} \right)^{1/2} \right).$$

Similarly, noting that the coefficients of E^u/L' are also contained in $\mathbb{Q}(j(E^u)) = L \subset L'$, Theorem 1.1 gives the following formula for the stable Faltings height,

$$h_{\text{Fal}}^{\text{stab}}(E/L) = h_{\text{Fal}}(E^u/L') = \log \left(N_{L'/\mathbb{Q}}(\Delta_{E^u/L'})^{1/12[L':\mathbb{Q}]} \left(\frac{\sqrt{|\Delta_3|}}{\pi} \right)^{1/2} \prod_{k=1}^8 \Gamma \left(\frac{k}{8} \right)^{-\chi_{-8}(k)/2} \right).$$

After expanding and using the fact that $N_{L'/\mathbb{Q}}(\Delta_{E^u/L'}) = 1$, we get

$$h_{\text{Fal}}^{\text{stab}}(E/L) = \log \left(\left(\frac{6\sqrt{2}}{\pi} \right)^{1/2} \left(\frac{\Gamma(5/8)\Gamma(7/8)}{\Gamma(1/8)\Gamma(3/8)} \right)^{1/2} \right).$$

Numerically, these values of the Faltings height are $h_{\text{Fal}}(E/L) \approx 6.22291129399367$ and $h_{\text{Fal}}^{\text{stab}}(E/L) \approx -0.721100481725771$.

3. TAYLOR EXPANSION OF THE NON-HOLOMORPHIC EISENSTEIN SERIES

Let \mathbb{H} denote the complex upper half-plane and define the stabilizer of the cusp ∞ by

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Then the non-holomorphic $SL_2(\mathbb{Z})$ Eisenstein series is defined by

$$E(z, s) := \sum_{M \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \text{Im}(Mz)^s, \quad z = x + iy \in \mathbb{H}, \quad \text{Re}(s) > 1.$$

Now, the Eisenstein series has the well-known Fourier expansion (see e.g. [Zag81, p. 278])

$$E(z, s) = y^s + \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)} y^{1-s} + \frac{4\pi^s}{\Gamma(s)\zeta(2s)} \sqrt{y} \sum_{n=1}^{\infty} \sigma_{1-2s}(n) n^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi ny) \cos(2\pi nx),$$

where $\Gamma(s)$ is Euler's Gamma function, $\zeta(s)$ is the Riemann zeta function, $\sigma_k(n) := \sum_{\ell|n} \ell^k$ is the k -divisor function, and K_ν is the K -Bessel function of order ν . The Fourier expansion shows that $E(z, s)$ extends to a meromorphic function on \mathbb{C} with a simple pole at $s = 1$.

We next make the shift $s \mapsto (s+1)/2$ in the Fourier expansion of $E(z, s)$ and calculate the Taylor expansion of the shifted Eisenstein series $E(z, (s+1)/2)$ at $s = -1$. For convenience, write

$$E\left(z, \frac{s+1}{2}\right) = A(z, s) + B(z, s) + C(z, s)$$

where

$$\begin{aligned} A(z, s) &:= y^{\frac{s+1}{2}}, \quad B(z, s) := \sqrt{\pi} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})} \frac{\zeta(s)}{\zeta(s+1)} y^{\frac{1-s}{2}}, \quad \text{and} \\ C(z, s) &:= \frac{4\pi^{\frac{s+1}{2}} \sqrt{y}}{\Gamma(\frac{s+1}{2})\zeta(s+1)} \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{\frac{s}{2}} K_{\frac{s}{2}}(2\pi ny) \cos(2\pi nx). \end{aligned}$$

Then the Taylor expansions of A, B and C at $s = -1$ are given as follows,

$$\begin{aligned} A(z, s) &= 1 + \log(\sqrt{y})(s+1) + O((s+1)^2), \\ B(z, s) &= -\frac{\pi}{6} y(s+1) + O((s+1)^2), \quad \text{and} \\ C(z, s) &= \left(-2 \sum_{n=1}^{\infty} \frac{\sigma_1(n) e^{-2\pi ny} \cos(2\pi nx)}{n} \right) (s+1) + O((s+1)^2). \end{aligned}$$

By combining these Taylor expansions, we get

$$E\left(z, \frac{s+1}{2}\right) = 1 + \left(\log(\sqrt{y}) - \frac{\pi}{6} y - 2 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{-2\pi ny} \cos(2\pi nx) \right) (s+1) + O((s+1)^2). \quad (3.1)$$

Next, recall that the *Dedekind eta function* is the weight $1/2$ modular form for $SL_2(\mathbb{Z})$ defined by the infinite product

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi iz}, \quad z \in \mathbb{H}.$$

One has the following identity relating the second term in the Taylor expansion of $E(z, (s+1)/2)$ at $s = -1$ to $\eta(z)$.

Proposition 3.1. *We have*

$$\log(\sqrt{y}|\eta(z)|^2) = \log(\sqrt{y}) - \frac{\pi}{6}y - 2 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{-2\pi ny} \cos(2\pi nx). \quad (3.2)$$

Proof. Define the complex-valued logarithm $\text{Log}(z) := \log|z| + i\text{Arg}(z)$ where $-\pi < \text{Arg}(z) \leq \pi$. Then noting that $\log|z| = \text{Re}(\text{Log}(z))$, we have

$$\begin{aligned} \log|\eta(z)| &= \text{Re}(\text{Log}(\eta(z))) = \text{Re}\left(\text{Log}\left(q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\right)\right) \\ &= \text{Re}(\text{Log}(q^{1/24})) + \text{Re}\left(\text{Log}\left(\prod_{n=1}^{\infty} (1 - q^n)\right)\right). \end{aligned}$$

Now, observe that

$$\text{Re}(\text{Log}(q^{1/24})) = \text{Re}\left(\log|q^{1/24}| + i\text{Arg}(q^{1/24})\right) = \log(e^{-\frac{\pi y}{12}}) = -\frac{\pi y}{12}.$$

Also, using the power series expansion

$$\text{Log}(1 - z) = - \sum_{m=1}^{\infty} \frac{z^m}{m}, \quad |z| < 1$$

we get

$$\begin{aligned} \text{Re}\left(\text{Log}\left(\prod_{n=1}^{\infty} (1 - q^n)\right)\right) &= \text{Re}\left(\sum_{n=1}^{\infty} \text{Log}(1 - q^n)\right) \\ &= -\text{Re}\left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q^{mn}}{m}\right) \\ &= -\text{Re}\left(\sum_{\ell=1}^{\infty} \frac{\sigma_1(\ell)}{\ell} q^{\ell}\right) \\ &= -\text{Re}\left(\sum_{\ell=1}^{\infty} \frac{\sigma_1(\ell)}{\ell} e^{2\pi i \ell x} e^{-2\pi \ell y}\right) \\ &= -\text{Re}\left(\sum_{\ell=1}^{\infty} \frac{\sigma_1(\ell)}{\ell} (\cos(2\pi \ell x) + i \sin(2\pi \ell x)) e^{-2\pi \ell y}\right) \\ &= - \sum_{\ell=1}^{\infty} \frac{\sigma_1(\ell)}{\ell} e^{-2\pi \ell y} \cos(2\pi \ell x). \end{aligned}$$

Then combining these calculations yields

$$\log(\sqrt{y}|\eta(z)|^2) = \log(\sqrt{y}) + 2 \log|\eta(z)| = \log(\sqrt{y}) - \frac{\pi}{6}y - 2 \sum_{\ell=1}^{\infty} \frac{\sigma_1(\ell)}{\ell} e^{-2\pi \ell y} \cos(2\pi \ell x). \quad \square$$

Finally, by combining (3.1) and (3.2), we arrive at the “renormalized” Kronecker limit formula

$$E\left(z, \frac{s+1}{2}\right) = 1 + \log(F(z))(s+1) + O((s+1)^2), \quad (3.3)$$

where we have defined

$$F(z) := \sqrt{\text{Im}(z)}|\eta(z)|^2. \quad (3.4)$$

Observe that the function $F(z)$ is $SL_2(\mathbb{Z})$ -invariant.

4. ZETA FUNCTIONS OF ORDERS AND CM VALUES OF EISENSTEIN SERIES

We begin by recalling some facts regarding orders in imaginary quadratic fields (see e.g. Cox [Cox13, §7]). Let K be an imaginary quadratic field of discriminant D . Given $f \in \mathbb{Z}^+$, let \mathcal{O}_f be the (unique) order of conductor f in K . A fractional \mathcal{O}_f -ideal \mathfrak{a} is a subset of K which is a non-zero finitely generated \mathcal{O}_f -module. A fractional \mathcal{O}_f -ideal \mathfrak{a} is *proper* if

$$\mathcal{O}_f = \{\beta \in K : \beta\mathfrak{a} \subset \mathfrak{a}\}.$$

It is known that a fractional \mathcal{O}_f -ideal is invertible if and only if it is proper (see [Cox13, Proposition 7.4]). Accordingly, let $I(\mathcal{O}_f)$ be the group of proper fractional \mathcal{O}_f -ideals, and let $P(\mathcal{O}_f)$ be the subgroup of $I(\mathcal{O}_f)$ consisting of principal fractional \mathcal{O}_f -ideals. The *ideal class group* of \mathcal{O}_f is defined as the quotient group

$$\text{Cl}(\mathcal{O}_f) := I(\mathcal{O}_f)/P(\mathcal{O}_f).$$

Let $h(\mathcal{O}_f) = \#\text{Cl}(\mathcal{O}_f)$ be the class number of \mathcal{O}_f .

The *Dedekind zeta function* of \mathcal{O}_f is defined by

$$\zeta_{\mathcal{O}_f}(s) := \sum_{\substack{\mathfrak{a} \in I(\mathcal{O}_f) \\ \mathfrak{a} \subset \mathcal{O}_f}} \frac{1}{N(\mathfrak{a})^s}, \quad \text{Re}(s) > 1.$$

Similarly, given an ideal class $A \in \text{Cl}(\mathcal{O}_f)$, we define the *ideal class zeta function* by

$$\zeta_A(s) := \sum_{\substack{I \in A \\ I \subset \mathcal{O}_f}} \frac{1}{N(I)^s}, \quad \text{Re}(s) > 1.$$

Then we have the decomposition

$$\zeta_{\mathcal{O}_f}(s) = \sum_{A \in \text{Cl}(\mathcal{O}_f)} \zeta_A(s).$$

Now, the discriminant of \mathcal{O}_f is given by $\Delta_f = f^2 D$. By [Cox13, Theorem 7.7], we may choose a proper integral ideal $\mathfrak{a} \in A$ with

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z} \left(\frac{-b + \sqrt{\Delta_f}}{2} \right)$$

where $[a, b, c](X, Y) = aX^2 + bXY + cY^2$ is a quadratic form of discriminant $b^2 - 4ac = \Delta_f$ with $(a, b, c) = 1$ and $a = N(\mathfrak{a}) > 0$.

For $\alpha \in K$, let α' denote the image of α under the nontrivial automorphism of K . Then

$$\mathfrak{a}' = \mathbb{Z}a + \mathbb{Z} \left(\frac{b + \sqrt{\Delta_f}}{2} \right).$$

Moreover, by [Cox13, equation (7.6)] we have $\mathfrak{a}^{-1} = \frac{1}{a}\mathfrak{a}'$, and thus

$$\mathfrak{a}^{-1} = \mathbb{Z} + \mathbb{Z} \left(\frac{b + \sqrt{\Delta_f}}{2a} \right) = \mathbb{Z} + \mathbb{Z}z_{\mathfrak{a}^{-1}} \tag{4.1}$$

where

$$z_{\mathfrak{a}^{-1}} := \frac{b + \sqrt{\Delta_f}}{2a} \in \mathbb{H}$$

is the root in the complex upper half-plane of the dehomogenized form $[a, -b, c](X, 1) = aX^2 - bX + c$.

Let \mathcal{O}_f^\times be the group of units in \mathcal{O}_f , and let $w_f = \#\mathcal{O}_f^\times$.

Proposition 4.1. *With notation as above, we have*

$$\zeta_{[\mathfrak{a}]}(s) = \frac{2}{w_f} \left(\frac{\sqrt{|\Delta_f|}}{2} \right)^{-s} \zeta(2s) E(z_{\mathfrak{a}^{-1}}, s).$$

We will need the following lemma.

Lemma 4.2. *Let \mathfrak{a} be a proper fractional \mathcal{O}_f -ideal. Then the map*

$$\phi : (\mathfrak{a}^{-1} \setminus \{0\}) / \mathcal{O}_f^\times \longrightarrow \{I \in [\mathfrak{a}] : I \subset \mathcal{O}_f\}$$

defined by $\phi([\alpha]) = \alpha\mathfrak{a}$ is a bijection.

Proof. We first prove that the map ϕ is well-defined. Observe that if $\alpha \in \mathfrak{a}^{-1}$, then $\alpha\mathfrak{a} \subseteq \mathcal{O}_f$ since $\mathfrak{a}^{-1}\mathfrak{a} = \mathcal{O}_f$. Next, observe that if $[\alpha] = [\beta]$, then $\alpha = \beta u$ for some unit $u \in \mathcal{O}_f^\times$. It follows that $\alpha\mathcal{O}_f = \beta u\mathcal{O}_f = \beta\mathcal{O}_f$, and hence $\alpha\mathfrak{a} = \beta\mathfrak{a}$. This proves that ϕ is well-defined.

To prove that ϕ is injective, suppose that $\alpha\mathfrak{a} = \beta\mathfrak{a}$. Then $\alpha\mathfrak{a}\mathfrak{a}^{-1} = \beta\mathfrak{a}\mathfrak{a}^{-1}$, which implies that $\alpha\mathcal{O}_f = \beta\mathcal{O}_f$, or equivalently, that $[\alpha] = [\beta]$. This proves that ϕ is injective.

To prove that ϕ is surjective, suppose that $I \in [\mathfrak{a}]$ with $I \subset \mathcal{O}_f$. Then $I = \alpha\mathfrak{a}$ for some $\alpha \in K^\times$, or equivalently, $I\mathfrak{a}^{-1} = \alpha\mathcal{O}_f$. Since I is integral, we have $I\mathfrak{a}^{-1} \subset \mathfrak{a}^{-1}$, so that $\alpha \in \mathfrak{a}^{-1}$. Then $[\alpha] \in (\mathfrak{a}^{-1} \setminus \{0\}) / \mathcal{O}_f^\times$ with $\phi([\alpha]) = \alpha\mathfrak{a} = I$. This proves that ϕ is surjective. \square

We now prove Proposition 4.1.

Proof of Proposition 4.1: Using Lemma 4.2 and (4.1), we get

$$\begin{aligned} \zeta_{[\mathfrak{a}]}(s) &= \sum_{\substack{I \in [\mathfrak{a}] \\ I \subset \mathcal{O}_f}} \frac{1}{N(I)^s} = \sum_{0 \neq \alpha \in \mathfrak{a}^{-1} / \mathcal{O}_f^\times} \frac{1}{N(\alpha\mathfrak{a})^s} \\ &= \frac{1}{N(\mathfrak{a})^s} \sum_{0 \neq \alpha \in \mathfrak{a}^{-1} / \mathcal{O}_f^\times} \frac{1}{N(\alpha)^s} \\ &= \frac{1}{a^s} \sum_{0 \neq \alpha \in \mathfrak{a}^{-1} / \mathcal{O}_f^\times} \frac{1}{|\alpha|^{2s}} \\ &= \frac{1}{w_f a^s} \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{|m + nz_{\mathfrak{a}^{-1}}|^{2s}} \\ &= \frac{1}{w_f} \left(\frac{\sqrt{|\Delta_f|}}{2} \right)^{-s} \left(\frac{\sqrt{|\Delta_f|}}{2a} \right)^s \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{|m + nz_{\mathfrak{a}^{-1}}|^{2s}} \\ &= \frac{1}{w_f} \left(\frac{\sqrt{|\Delta_f|}}{2} \right)^{-s} \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{\text{Im}(z_{\mathfrak{a}^{-1}})^s}{|m + nz_{\mathfrak{a}^{-1}}|^{2s}}. \end{aligned}$$

Now, define the quadratic form

$$Q_z(m, n) := \frac{|m + nz|^2}{\text{Im}(z)}, \quad z \in \mathbb{H}$$

and the associated *Epstein zeta function*

$$Z(Q_z, s) := \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{Q_z(m, n)^s}, \quad \text{Re}(s) > 1. \quad (4.2)$$

Then one has the following well-known identity (see e.g. [Zag81, equation (2)]),

$$Z(Q_z, s) = 2\zeta(2s)E(z, s). \quad (4.3)$$

Finally, using (4.2) and (4.3), it follows that

$$\sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{\operatorname{Im}(z_{\mathfrak{a}^{-1}})^s}{|m + nz_{\mathfrak{a}^{-1}}|^{2s}} = Z(Q_{z_{\mathfrak{a}^{-1}}}, s) = 2\zeta(2s)E(z_{\mathfrak{a}^{-1}}, s).$$

□

5. A CHOWLA-SELBERG FORMULA FOR IMAGINARY QUADRATIC ORDERS

In this section we will prove the following theorem.

Theorem 5.1. *With notation as in Section 4, we have*

$$\prod_{[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}}) = \left(\frac{1}{4\pi\sqrt{|\Delta_f|}} \right)^{h(\mathcal{O}_f)/2} \frac{|D|}{\prod_{k=1}^{|D|} \Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)w_D h(\mathcal{O}_f)/4h(D)}} \prod_{p|f} p^{-e(p)h(\mathcal{O}_f)/2},$$

where $F(z)$ is defined in (3.4), $z_{\mathfrak{a}^{-1}}$ is a CM point as in (4.1), and

$$e(p) := \frac{(1 - p^{\operatorname{ord}_p(f)})(1 - \chi_D(p))}{p^{\operatorname{ord}_p(f)-1}(1-p)(\chi_D(p) - p)}.$$

Before proving Theorem 5.1, we demonstrate how it can be used to explicitly evaluate a CM value of $\eta(z)$. We then numerically verify the resulting identity using **SageMath** [S⁺09].

Example 5.2. Let $K = \mathbb{Q}(i)$, and consider the order of conductor $f = 2$ in K , i.e., the non-maximal order $\mathcal{O}_2 = \mathbb{Z} + 2\mathbb{Z}[i] = \mathbb{Z} + \mathbb{Z}2i$. Since the discriminant of K is $D = -4$, the discriminant of the order is $\Delta_2 = 2^2(-4) = -16$, and also $h(-4) = 1$ and $w_{-4} = 4$. Using **SageMath**, we find that $h(\mathcal{O}_2) = 1$, and hence $\operatorname{Cl}(\mathcal{O}_2) = \{[\mathcal{O}_2]\}$. Since $\mathcal{O}_2^{-1} = \mathcal{O}_2 = \mathbb{Z} + \mathbb{Z}2i$, from (4.1) we can take $z_{\mathcal{O}_2^{-1}} = 2i$ for the CM point. It follows that

$$\prod_{[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_2)} F(z_{\mathfrak{a}^{-1}}) = F(z_{\mathcal{O}_2^{-1}}) = F(2i) = \sqrt{|\operatorname{Im}(2i)|} |\eta(2i)|^2 = \sqrt{2} |\eta(2i)|^2.$$

On the other hand, we have

$$\left(\frac{1}{4\pi\sqrt{|\Delta_2|}} \right)^{h(\mathcal{O}_2)/2} \prod_{k=1}^4 \Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)w_{-4}h(\mathcal{O}_2)/4h(-4)} \prod_{p|2} p^{e(p)h(\mathcal{O}_2)/2} = \frac{1}{4\sqrt{\pi}} \prod_{k=1}^4 \Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)} 2^{e(2)/2}.$$

Therefore, by Theorem 5.1 we get

$$\sqrt{2} |\eta(2i)|^2 = \frac{1}{4\sqrt{\pi}} \prod_{k=1}^4 \Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)} 2^{e(2)/2}. \quad (5.1)$$

Now, the values of the Kronecker symbol are $\chi_{-4}(1) = 1$, $\chi_{-4}(2) = 0$, $\chi_{-4}(3) = -1$, and $\chi_{-4}(4) = 0$. Since $\chi_{-4}(2) = 0$, we see that $e(2) = 1/2$. Therefore, after expanding the product in (5.1) we get

$$\sqrt{2} |\eta(2i)|^2 = \frac{1}{2^{7/4}\pi^{1/2}} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)^{-1}. \quad (5.2)$$

Furthermore, using the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

with $z = 1/4$ yields

$$\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}\Gamma\left(\frac{1}{4}\right)^{-1}.$$

Then, substituting this into (5.2) gives

$$|\eta(2i)|^2 = \frac{1}{2^{11/4}\pi^{3/2}}\Gamma\left(\frac{1}{4}\right)^2.$$

Finally, observing that $\eta(2i)$ is a positive real number, we see that

$$\eta(2i) = \frac{1}{2^{11/8}\pi^{3/4}}\Gamma\left(\frac{1}{4}\right).$$

Using `SageMath`, one can check that both sides of the previous equality are approximately 0.592382781332416, which serves as a numerical verification of the identity in Theorem 5.1.

Proof of Theorem 5.1: By Proposition 4.1, we have

$$\zeta_{[\mathfrak{a}]}((s+1)/2) = \frac{2}{w_f} \left(\frac{\sqrt{|\Delta_f|}}{2} \right)^{-(s+1)/2} \zeta(s+1) E(z_{\mathfrak{a}^{-1}}, (s+1)/2).$$

Then summing over all ideal classes in $\text{Cl}(\mathcal{O}_f)$ yields

$$\zeta_{\mathcal{O}_f}((s+1)/2) = \frac{2}{w_f} \left(\frac{\sqrt{|\Delta_f|}}{2} \right)^{-(s+1)/2} \zeta(s+1) \sum_{[\mathfrak{a}] \in \text{Cl}(\mathcal{O}_f)} E(z_{\mathfrak{a}^{-1}}, (s+1)/2).$$

For convenience, define the function

$$g_{\mathcal{O}_f}(s) := \frac{w_f}{2} \left(\frac{\sqrt{|\Delta_f|}}{2} \right)^{(s+1)/2} \frac{\zeta_{\mathcal{O}_f}((s+1)/2)}{\zeta(s+1)},$$

so that

$$g_{\mathcal{O}_f}(s) = \sum_{[\mathfrak{a}] \in \text{Cl}(\mathcal{O}_f)} E(z_{\mathfrak{a}^{-1}}, (s+1)/2). \quad (5.3)$$

Then recalling the renormalized Kronecker limit formula (3.3), comparing Taylor expansions at $s = -1$ on both sides of (5.3) yields

$$g'_{\mathcal{O}_f}(-1) = \sum_{[\mathfrak{a}] \in \text{Cl}(\mathcal{O}_f)} \log(F(z_{\mathfrak{a}^{-1}})),$$

or equivalently,

$$\prod_{[\mathfrak{a}] \in \text{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}}) = \exp(g'_{\mathcal{O}_f}(-1)). \quad (5.4)$$

It remains to calculate $g'_{\mathcal{O}_f}(-1)$.

Our starting point is the factorization (see e.g. [AIK14, Proposition 10.18 (2)])

$$\zeta_{\mathcal{O}_f}(s) = \zeta(s) L_f(s) L(\chi_D, s),$$

where

$$L_f(s) := \prod_{p|f} \frac{(1-p^{-s})(1-\chi_D(p)p^{-s}) - p^{\text{ord}_p(f)(1-2s)-1}(1-p^{1-s})(\chi_D(p) - p^{1-s})}{1-p^{1-2s}}.$$

Then we may write

$$g_{\mathcal{O}_f}(s) = \frac{w_f}{2} \left(\frac{\sqrt{|\Delta_f|}}{2} \right)^{(s+1)/2} \frac{\zeta((s+1)/2)}{\zeta(s+1)} L_f((s+1)/2) L(\chi_D, (s+1)/2).$$

Now, a calculation with the product rule yields

$$g'_{\mathcal{O}_f}(-1) = \frac{w_f}{4} L_f(0) L(\chi_D, 0) \left(\log \left(\frac{\sqrt{|\Delta_f|}}{2} \right) - \frac{\zeta'(0)}{\zeta(0)} + \frac{L'(\chi_D, 0)}{L(\chi_D, 0)} + \frac{L'_f(0)}{L_f(0)} \right).$$

To further simplify this identity, we note that

$$L_f(0) = f \prod_{p|f} \left(1 - \frac{\chi_D(p)}{p} \right).$$

Then using Dirichlet's class number formula

$$L(\chi_D, 0) = 2h(D)/w_D, \tag{5.5}$$

the identity (see e.g. [Cox13, Theorem 7.24])

$$h(\mathcal{O}_f) = \frac{h(D)f}{[\mathcal{O}_K^\times : \mathcal{O}_f^\times]} \prod_{p|f} \left(1 - \frac{\chi_D(p)}{p} \right),$$

and $[\mathcal{O}_K^\times : \mathcal{O}_f^\times] = w_D/w_f$, we get

$$\frac{w_f}{4} L_f(0) L(\chi_D, 0) = \frac{h(\mathcal{O}_f)}{2}.$$

It follows that

$$g'_{\mathcal{O}_f}(-1) = \frac{h(\mathcal{O}_f)}{2} \left(\log \left(\frac{\sqrt{|\Delta_f|}}{2} \right) - \frac{\zeta'(0)}{\zeta(0)} + \frac{L'(\chi_D, 0)}{L(\chi_D, 0)} + \frac{L'_f(0)}{L_f(0)} \right). \tag{5.6}$$

We next evaluate the logarithmic derivatives of $\zeta(s)$, $L(\chi_D, s)$ and $L_f(s)$ at $s = 0$. First, using the special values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2} \log(2\pi)$, we get

$$\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi). \tag{5.7}$$

Next, we have the decomposition

$$L(\chi_D, s) = |D|^{-s} \sum_{k=1}^{|D|} \chi_D(k) \zeta \left(s, \frac{k}{|D|} \right), \tag{5.8}$$

where

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad x > 0, \quad \operatorname{Re}(s) > 1$$

is the Hurwitz zeta function. Lerch [Ler87] showed that

$$\zeta(s, x) = \frac{1}{2} - x + \log \left(\frac{\Gamma(x)}{\sqrt{2\pi}} \right) s + O(s^2), \tag{5.9}$$

where

$$\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t} dt$$

is Euler's gamma function. Then we substitute (5.9) into (5.8), differentiate, and use the class number formula (5.5) to get

$$\frac{L'(\chi_D, 0)}{L(\chi_D, 0)} = -\log(|D|) + \frac{w_D}{2h(D)} \sum_{k=1}^{|D|} \chi_D(k) \log \left(\Gamma \left(\frac{k}{|D|} \right) \right). \quad (5.10)$$

Finally, by Lemma 5.3 (which is stated and proved at the end of this section), we have

$$\frac{L'_f(0)}{L_f(0)} = \log \left(f^{-2} \prod_{p|f} p^{-e(p)} \right), \quad (5.11)$$

where

$$e(p) := \frac{(1 - p^{\text{ord}_p(f)})(1 - \chi_D(p))}{p^{\text{ord}_p(f)-1}(1-p)(\chi_D(p) - p)}.$$

Substituting the logarithmic derivatives (5.7), (5.10), and (5.11) into (5.6) and using $|\Delta_f| = f^2|D|$, we get

$$\begin{aligned} g'_{\mathcal{O}_f}(-1) &= \frac{h(\mathcal{O}_f)}{2} \log \left(\left(\frac{\sqrt{|\Delta_f|}}{4\pi f^2 |D|} \right) \prod_{k=1}^{|D|} \Gamma \left(\frac{k}{|D|} \right)^{\chi_D(k)w_D/2h(D)} \prod_{p|f} p^{-e(p)} \right) \\ &= \log \left(\left(\frac{1}{4\pi \sqrt{|\Delta_f|}} \right)^{h(\mathcal{O}_f)/2} \prod_{k=1}^{|D|} \Gamma \left(\frac{k}{|D|} \right)^{\chi_D(k)w_D h(\mathcal{O}_f)/4h(D)} \prod_{p|f} p^{-e(p)h(\mathcal{O}_f)/2} \right). \end{aligned}$$

Therefore,

$$\exp(g'_{\mathcal{O}_f}(-1)) = \left(\frac{1}{4\pi \sqrt{|\Delta_f|}} \right)^{h(\mathcal{O}_f)/2} \prod_{k=1}^{|D|} \Gamma \left(\frac{k}{|D|} \right)^{\chi_D(k)w_D h(\mathcal{O}_f)/4h(D)} \prod_{p|f} p^{-e(p)h(\mathcal{O}_f)/2},$$

which by virtue of (5.4) proves the theorem. \square

It remains to prove the following lemma.

Lemma 5.3. *We have*

$$\frac{L'_f(0)}{L_f(0)} = \log \left(f^{-2} \prod_{p|f} p^{-e(p)} \right),$$

where

$$e(p) := \frac{(1 - p^{\text{ord}_p(f)})(1 - \chi_D(p))}{p^{\text{ord}_p(f)-1}(1-p)(\chi_D(p) - p)}.$$

Proof. Define the functions

$$G_p(s) := (1 - p^{-s})(1 - \chi_D(p)p^{-s}) - p^{\text{ord}_p(f)(1-2s)-1}(1 - p^{1-s})(\chi_D(p) - p^{1-s})$$

and

$$H_p(s) := 1 - p^{1-2s}.$$

Then $L_f(s)$ can be written as

$$L_f(s) = \prod_{p|f} \frac{G_p(s)}{H_p(s)}.$$

Now, we have

$$\begin{aligned}
\frac{L'_f(s)}{L_f(s)} &= \frac{d}{ds} \log(L_f(s)) = \frac{d}{ds} \log \left(\prod_{p|f} \frac{G_p(s)}{H_p(s)} \right) \\
&= \frac{d}{ds} \sum_{p|f} \log \left(\frac{G_p(s)}{H_p(s)} \right) \\
&= \sum_{p|f} \left(\frac{d}{ds} \log(G_p(s)) - \frac{d}{ds} \log(H_p(s)) \right) \\
&= \sum_{p|f} \left(\frac{G'_p(s)}{G_p(s)} - \frac{H'_p(s)}{H_p(s)} \right).
\end{aligned}$$

Therefore, it suffices to evaluate the logarithmic derivatives of $G_p(s)$ and $H_p(s)$ at $s = 0$. Since

$$H'_p(s) = 2 \log(p) p^{1-2s},$$

we get

$$\frac{1}{\log(p)} \frac{H'_p(0)}{H_p(0)} = \frac{2p}{1-p}.$$

Next, a calculation with the product rule yields

$$\begin{aligned}
G'_p(s) &= (1 - p^{-s}) \log(p) \chi_D(p) p^{-s} + (1 - \chi_D(p) p^{-s}) \log(p) p^{-s} \\
&\quad + 2 \operatorname{ord}_p(f) \log(p) p^{\operatorname{ord}_p(f)(1-2s)-1} (1 - p^{1-s}) (\chi_D(p) - p^{1-s}) \\
&\quad - p^{\operatorname{ord}_p(f)(1-2s)-1} (\chi_D(p) - p^{1-s}) \log(p) p^{1-s} \\
&\quad - p^{\operatorname{ord}_p(f)(1-2s)-1} (1 - p^{1-s}) \log(p) p^{1-s},
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{\log(p)} \frac{G'_p(0)}{G_p(0)} &= \\
&= \frac{1 - \chi_D(p) + 2 \operatorname{ord}_p(f) p^{\operatorname{ord}_p(f)-1} (1 - p) (\chi_D(p) - p) - p^{\operatorname{ord}_p(f)} (\chi_D(p) - p) - p^{\operatorname{ord}_p(f)} (1 - p)}{-p^{\operatorname{ord}_p(f)-1} (1 - p) (\chi_D(p) - p)}.
\end{aligned}$$

Combining the preceding calculations gives

$$\begin{aligned}
&\frac{1}{\log(p)} \left(\frac{G'_p(0)}{G_p(0)} - \frac{H'_p(0)}{H_p(0)} \right) \\
&= \frac{1 - \chi_D(p) + 2 \operatorname{ord}_p(f) p^{\operatorname{ord}_p(f)-1} (1 - p) (\chi_D(p) - p) - p^{\operatorname{ord}_p(f)} (\chi_D(p) - p) - p^{\operatorname{ord}_p(f)} (1 - p)}{-p^{\operatorname{ord}_p(f)-1} (1 - p) (\chi_D(p) - p)} - \frac{2p}{1-p} \\
&= -2 \operatorname{ord}_p(f) + \frac{1 - \chi_D(p) - p^{\operatorname{ord}_p(f)} (\chi_D(p) - p) - p^{\operatorname{ord}_p(f)} (1 - p) + 2p^{\operatorname{ord}_p(f)} (\chi_D(p) - p)}{-p^{\operatorname{ord}_p(f)-1} (1 - p) (\chi_D(p) - p)} \\
&= -2 \operatorname{ord}_p(f) - \frac{(1 - p^{\operatorname{ord}_p(f)}) (1 - \chi_D(p))}{p^{\operatorname{ord}_p(f)-1} (1 - p) (\chi_D(p) - p)}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\frac{L'_f(0)}{L_f(0)} &= \sum_{p|f} -\log(p) \left(2 \operatorname{ord}_p(f) + \frac{(1 - p^{\operatorname{ord}_p(f)})(1 - \chi_D(p))}{p^{\operatorname{ord}_p(f)-1}(1-p)(\chi_D(p) - p)} \right) \\
&= \log \left(\prod_{p|f} p^{-\left(2 \operatorname{ord}_p(f) + \frac{(1 - p^{\operatorname{ord}_p(f)})(1 - \chi_D(p))}{p^{\operatorname{ord}_p(f)-1}(1-p)(\chi_D(p) - p)} \right)} \right) \\
&= \log \left(\left(\prod_{p|f} p^{\operatorname{ord}_p(f)} \right)^{-2} \prod_{p|f} p^{-e(p)} \right) \\
&= \log \left(f^{-2} \prod_{p|f} p^{-e(p)} \right).
\end{aligned}$$

□

6. FALTINGS HEIGHTS OF CM ELLIPTIC CURVES AND THE PROOF OF THEOREM 1.1

We first recall the definition of the Faltings height of an elliptic curve (see e.g. [Sil86] and [Mil08, Chapter 26]). Let L be a number field with ring of integers \mathcal{O}_L . Let E/L be an elliptic curve over L and let $\mathcal{E}/\mathcal{O}_L$ be a Néron model for E/L . Then the *Faltings height* of E/L is defined by

$$h_{\text{Fal}}(E/L) := \frac{1}{[L : \mathbb{Q}]} \deg(\omega_{\mathcal{E}/\mathcal{O}_L}),$$

where $\omega_{\mathcal{E}/\mathcal{O}_L} = s^* \Omega_{\mathcal{E}/\mathcal{O}_L}$ is the metrized line bundle on $\operatorname{Spec}(\mathcal{O}_L)$ given by the pullback of the sheaf of Néron differentials $\Omega_{\mathcal{E}/\mathcal{O}_L}$ by the zero section $s : \operatorname{Spec}(\mathcal{O}_L) \rightarrow \mathcal{E}$.

The Faltings height can be given more explicitly as follows. Given a differential $\omega \in H^0(E/L, \Omega_{E/L})$, we have

$$h_{\text{Fal}}(E/L) = \frac{\log(\#(\Omega_{E/L}/\mathcal{O}_L\omega))}{[L : \mathbb{Q}]} - \frac{1}{2[L : \mathbb{Q}]} \sum_{\sigma: L \rightarrow \mathbb{C}} \log \left(\frac{i}{2} \int_{E^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma} \right).$$

Now, recall that $\Delta_{E/L}$ is the minimal discriminant of E/L and $j(E)$ is the j -invariant of E/L . The following proposition is based on a result of Silverman [Sil86, Proposition 1.1].

Proposition 6.1. *Suppose that E/L has complex multiplication by an order \mathcal{O}_f in an imaginary quadratic field K . Then if the Weierstrass equation for E has coefficients in $\mathbb{Q}(j(E)) \subset L$, we have*

$$h_{\text{Fal}}(E/L) = \frac{\log(N_{L/\mathbb{Q}}(\Delta_{E/L}))}{12[L : \mathbb{Q}]} - \log(2\pi) - \frac{1}{h(\mathcal{O}_f)} \sum_{[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)} \log(F(z_{\mathfrak{a}^{-1}})),$$

where $F(z)$ is defined in equation (3.4) and $z_{\mathfrak{a}^{-1}}$ is a CM point as in equation (4.1).

Proof. Given $\sigma \in \operatorname{Hom}(L, \mathbb{C})$, let $z_\sigma \in \mathbb{H}$ be a complex number such that

$$E^\sigma(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}z_\sigma). \quad (6.1)$$

Moreover, let $\Delta(z) := (2\pi)^{12} \eta(z)^{24}$ be the discriminant function. Then Silverman [Sil86, Proposition 1.1] proved that the Faltings height of E/L is given by

$$h_{\text{Fal}}(E/L) = \frac{\log(N_{L/\mathbb{Q}}(\Delta_{E/L}))}{12[L : \mathbb{Q}]} - \frac{1}{12[L : \mathbb{Q}]} \sum_{\sigma \in \operatorname{Hom}(L, \mathbb{C})} \log(\operatorname{Im}(z_\sigma)^6 |\Delta(z_\sigma)|). \quad (6.2)$$

Since

$$\frac{1}{12} \log(\operatorname{Im}(z_\sigma)^6 |\Delta(z_\sigma)|) = \log(2\pi) + \log(F(z_\sigma)),$$

equation (6.2) becomes

$$h_{\text{Fal}}(E/L) = \frac{\log(N_{L/\mathbb{Q}}(\Delta_{E/L}))}{12[L:\mathbb{Q}]} - \log(2\pi) - \frac{1}{[L:\mathbb{Q}]} \sum_{\sigma \in \operatorname{Hom}(L, \mathbb{C})} \log(F(z_\sigma)). \quad (6.3)$$

Now, write

$$\sum_{\sigma \in \operatorname{Hom}(L, \mathbb{C})} \log(F(z_\sigma)) = \sum_{\tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})} \sum_{\substack{\sigma \in \operatorname{Hom}(L, \mathbb{C}) \\ \sigma|_{\mathbb{Q}(j(E))} = \tau}} \log(F(z_\sigma)).$$

Since E/L has coefficients in $\mathbb{Q}(j(E))$, for each fixed $\tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})$ we can take the same $z_\sigma \in \mathbb{H}$ in the isomorphism (6.1) for all $\sigma \in \operatorname{Hom}(L, \mathbb{C})$ such that $\sigma|_{\mathbb{Q}(j(E))} = \tau$. Therefore, if we let $\sigma_\tau \in \operatorname{Hom}(L, \mathbb{C})$ denote any of the $[L:\mathbb{Q}(j(E))]$ embeddings which extend $\tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})$, we have

$$\sum_{\tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})} \sum_{\substack{\sigma \in \operatorname{Hom}(L, \mathbb{C}) \\ \sigma|_{\mathbb{Q}(j(E))} = \tau}} \log(F(z_\sigma)) = \sum_{\tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})} [L:\mathbb{Q}(j(E))] \log(F(z_{\sigma_\tau})).$$

By Shimura [Shi94, Theorem 7.6], we have $[\mathbb{Q}(j(E)):\mathbb{Q}] = h(\mathcal{O}_f)$ and

$$\{j(E)^\tau : \tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})\} = \{j(\mathfrak{a}^{-1}) : [\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)\}.$$

Then for each $\tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})$, there is a unique $[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)$ such that $E^{\sigma_\tau}(\mathbb{C}) \cong \mathbb{C}/\mathfrak{a}^{-1}$. Recalling that $\mathfrak{a}^{-1} = \mathbb{Z} + \mathbb{Z}z_{\mathfrak{a}^{-1}}$, we get

$$\mathbb{C}/(\mathbb{Z} + \mathbb{Z}z_{\sigma_\tau}) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}z_{\mathfrak{a}^{-1}}),$$

and thus the points z_{σ_τ} and $z_{\mathfrak{a}^{-1}}$ are $SL_2(\mathbb{Z})$ -equivalent (see e.g. [Sil94, Proposition I.4.4]). Since $F(z)$ is $SL_2(\mathbb{Z})$ -invariant, it follows that

$$\sum_{\tau \in \operatorname{Hom}(\mathbb{Q}(j(E)), \mathbb{C})} [L:\mathbb{Q}(j(E))] \log(F(z_{\sigma_\tau})) = \frac{[L:\mathbb{Q}]}{h(\mathcal{O}_f)} \sum_{[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)} \log(F(z_{\mathfrak{a}^{-1}})).$$

Finally, the preceding calculations yield

$$\frac{1}{[L:\mathbb{Q}]} \sum_{\sigma \in \operatorname{Hom}(L, \mathbb{C})} \log(F(z_\sigma)) = \frac{1}{h(\mathcal{O}_f)} \sum_{[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)} \log(F(z_{\mathfrak{a}^{-1}})),$$

which by virtue of (6.3) completes the proof. \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1: By Proposition 6.1, we have

$$h_{\text{Fal}}(E/L) = \log \left(N_{L/\mathbb{Q}}(\Delta_{E/L})^{1/12[L:\mathbb{Q}]} (2\pi)^{-1} \prod_{[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}})^{-1/h(\mathcal{O}_f)} \right). \quad (6.4)$$

Moreover, by Theorem 5.1 we have

$$\prod_{[\mathfrak{a}] \in \operatorname{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}})^{-1/h(\mathcal{O}_f)} = \left(\frac{1}{4\pi\sqrt{|\Delta_f|}} \right)^{-1/2} \prod_{k=1}^{|D|} \Gamma \left(\frac{k}{|D|} \right)^{-\chi_D(k)w_D/4h(D)} \prod_{p|f} p^{e(p)/2}. \quad (6.5)$$

Then by substituting (6.5) into (6.4) and simplifying, we obtain Theorem 1.1. \square

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