ON THE ZEROES OF HALF-INTEGRAL WEIGHT EISENSTEIN SERIES ON $\Gamma_0(4)$

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1. OVERVIEW

Let $k \geq 5$ be an odd integer. In this paper we study the zeroes of the weight $\frac{k}{2}$ Eisenstein series, $E_\infty(z)$, of $\Gamma_0(4)$, which is defined as

$$(1) \qquad E_\infty(z) = e^{\frac{\pi i k}{4}} \sum_{\substack{(d,2c)=1 \\ c>0}} \frac{G\left(\frac{-d}{4c}\right)^k}{(4cz+d)^{\frac{k}{2}}},$$

where $G\left(\frac{m}{n}\right) = \frac{g\left(\frac{m}{n}\right)}{|g\left(\frac{m}{n}\right)|}$ and $g\left(\frac{m}{n}\right) = \sum_{a \bmod n} e^{\frac{2\pi i a^2 m}{n}}$ is the Gauss sum. Note that, though $E_\infty(z)$ depends on k as well as z, we omit this dependence from the notation.

This Eisenstein series is interesting to study due to its weight; many authors have studied the zeroes of integral weight modular forms, including [5] and [7]. On the other hand, there have been studies of half integral weight modular forms (such as that of [1]), but none of these have been on the zeroes of their Eisenstein series. As the simplest half integral weight modular form, $E_\infty(z)$ is thus a natural choice to study.

Before stating our results, we first state some basic properties of $\Gamma_0(4)$ and its Eisenstein series. Many of these properties can be found in [2] and [4].

In order to study the zeroes of E_∞ on all of $\mathbb{H}$, it suffices to find the zeroes of $E_\infty(z)$ in a fundamental domain. Our choice of fundamental domain, F_∞, of $\Gamma_0(4)$ is the union of six regions [3], as shown in Figure 1.

There are 3 cusps of F_∞: $0, \frac{1}{2}$, and ∞. We have already seen the Eisenstein series associated to the cusp ∞. The other two Eisenstein series are defined as

$$(2) \qquad E_0(z) = \sum_{\substack{(u,2v)=1 \\ u>0}} \frac{\left(\frac{-v}{u}\right)\epsilon_u^k}{(uz+v)^{\frac{k}{2}}}, \qquad\qquad E_{\frac{1}{2}}(z) = e^{\frac{-\pi i k}{4}} \sum_{\substack{(2c,d)=1 \\ d>0}} \frac{G\left(\frac{d-2c}{8d}\right)^k}{(dz+c)^{\frac{k}{2}}}.$$
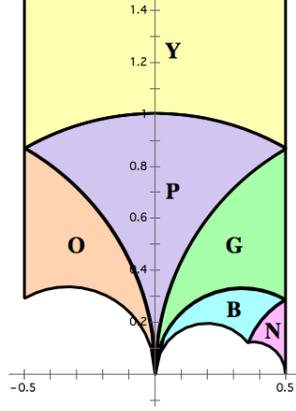
FIGURE 1. F_∞

Here $\left(\frac{a}{b}\right)$ is the Jacobi symbol and

$$\epsilon_u = \begin{cases} 1, & u \equiv 1 \mod 4 \\ i, & u \equiv 3 \mod 4. \end{cases}$$

We will also find it useful to use the Fourier expansion of $E_0(z)$ (see [4]), which is

$$(3) \qquad E_0(z) = 2^{\frac{k}{2}} \sum_{l=1}^{\infty} b_\ell q^\ell$$

where $q = e^{2\pi i z}$, and

$$(4) \qquad b_\ell = \frac{\pi^{\frac{k}{2}} \ell^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2}) e^{\frac{\pi i k}{4}}} \sum_{n>0 \ odd} \epsilon_n^k n^{-\frac{k}{2}} \sum_{j=0}^{n-1} \left(\frac{j}{n}\right) e^{-\frac{2\pi i \ell j}{n}}.$$

The following relations exist between $E_\infty(z)$ and $E_0(z)$ and $E_{\frac{1}{2}}(z)$:

$$(5) \qquad E_0\left(-\frac{1}{4z}\right) = (4z)^{\frac{k}{2}} i^{-k} E_\infty(z),$$

$$(6) \qquad E_{\frac{1}{2}}\left(z + \frac{1}{2}\right) = 2^{\frac{k}{2}} (2z+1)^{-\frac{k}{2}} E_\infty\left(\frac{z}{2z+1}\right).$$

These allow us to study zeroes of $E_\infty(z)$ near the cusps 0 and $\frac{1}{2}$ by studying zeroes of $E_0(z)$ and $E_{\frac{1}{2}}(z)$ in regions with $\text{Im}(z) \geq \frac{1}{2}$. We study $E_0(z)$ on F_0 and $E_{\frac{1}{2}}(z)$ on $F_{\frac{1}{2}}$, where F_0 and $F_{\frac{1}{2}}$ are the relevant fundamental domains that correspond to F_∞. These are shown in Figure 2.

Based on the studies by [5] and [7], one might guess that the zeroes of $E_\infty(z)$ would lie along the boundaries of the fundamental domains. In our case, we have the following conjecture:
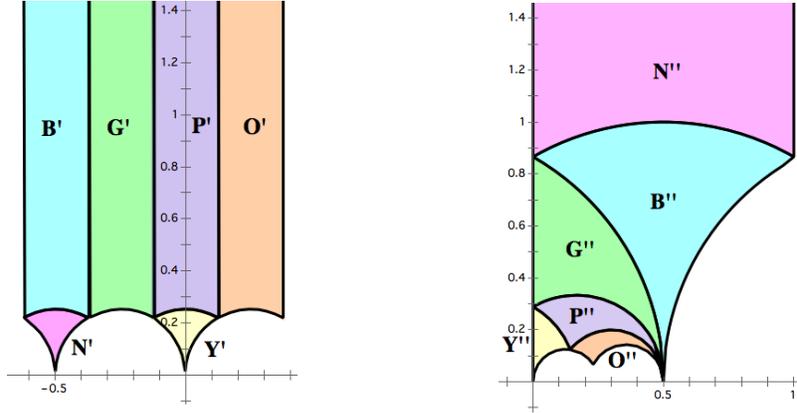
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FIGURE 2. F_0 (left) and $F_{\frac{1}{2}}$ (right)

Conjecture 1.1. *All but one of one of the zeroes of $E_\infty(z)$ in F_∞ lie on the circle $|z - \frac{1}{4}| = \frac{1}{4}$, which is within the regions N and B. The final zero lies in P along the line $x = 0$.*

It is worth noting that, though these zeroes are not on the boundaries of F_∞, it is possible to choose a fundamental domain in which this is the case (see [1]). Also, a more technical note: in Conjecture 1.1 and throughout the paper, we use the term zeroes to refer specifically to the non–trivial zeroes; here, the trivial zeroes are those at the cusps 0 and $\frac{1}{2}$.

This paper will not address the zero in P. To study the zeroes in the regions N and B, we study the corresponding zeroes of $E_0(z)$ and $E_{\frac{1}{2}}(z)$. More specifically, the zeroes in these regions lie on the line $x = -\frac{1}{2}$ in N' and B' and the line $x = \frac{1}{2}$ in N'' and B'', which correspond to the circle $|z - \frac{1}{4}| = \frac{1}{4}$ in F_∞. For example, Figure 3 shows the zeroes in each fundamental domain when $k = 15$. The only zero not seen in any of the pictures is the zero in P; this is because in each of the pictures it is too low or to high to be seen. For instance, the zero is located at $\approx 5.881i$ in F_0.
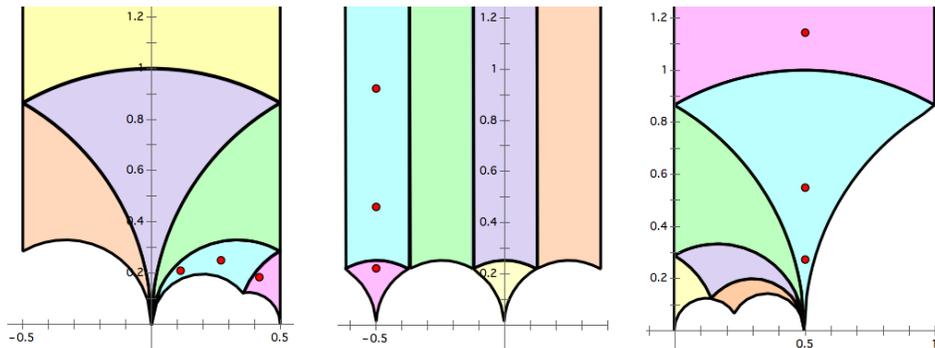

FIGURE 3. zeroes for $k = 15$

As such, in the following sections we consider $E_0(-\frac{1}{2} + iy)$ and $E_{\frac{1}{2}}(\frac{1}{2} + iy)$. In order to restrict our studies to regions with not–too–small imaginary part, we utilize the following additional automorphy relations:

$$(7) \qquad\qquad\qquad E_{\frac{1}{2}}(z) = i^k E_{\frac{1}{2}}(z + 1),$$

$$(8) \qquad\qquad\qquad E_{\frac{1}{2}}(z) = (2z + 1)^{-\frac{k}{2}} E_0\left(\frac{z}{2z + 1}\right).$$

By relation (8) and the periodicity of $E_0(z)$, $E_{\frac{1}{2}}(\frac{1}{2} + iy)$ where $y \geq \frac{1}{2}$ corresponds to $E_0(-\frac{1}{2} + iy')$ where $y' \leq \frac{1}{2}$. As such, it suffices to study only $y \geq \frac{1}{2}$ in each case so as to count each zero only once.

Theorem 1.2. *For k sufficiently large, $E_0(-\frac{1}{2} + iy)$ and $E_{\frac{1}{2}}(\frac{1}{2} + iy)$ each have at least $\frac{k}{8} - O(\sqrt{k \log k})$ zeroes for $y \geq \frac{1}{2}$.*

Proving this theorem will be the focus of the rest of the paper. The following corollary relates this theorem to zeroes of $E_\infty(z)$ and makes considerable progress toward proving Conjecture 1.1.

Corollary 1.3. *For k sufficiently large, all but at most $O(\sqrt{k \log k})$ zeroes of $E_\infty(z)$ lie on the circle $|z - \frac{1}{4}| = \frac{1}{4}$ in F_∞.*

Note that this corollary shows asymptotically that we have found all of the zeroes on the circle.

Proof. The valence formula [4] for $\Gamma_0(4)$ tells us that, at most, $E_\infty(z)$ has $\lfloor \frac{k}{4} \rfloor$ zeroes (here, some care must be taken to count zeroes at the cusps with appropriate weights). Thus, the corollary will follow directly from Theorem 1.2. $\qquad\square$

To prove Theorem 1.2, we need a few simpler results. First, we prove that $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy)$ is a real–valued function. We then find a real–valued trigonometric approximation of $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy)$, which we denote as M_0. We bound the error of this approximation for large k and $y \leq \frac{\sqrt{k}}{\sqrt{2 \log k}}$, . Similarly, we then prove that $E_{\frac{1}{2}}(\frac{1}{2} + iy)$ is a real–valued function with real–valued trigonometric approximation $M_{\frac{1}{2}}$. We again bound this error under the same restrictions for y and k.

In the final section, we use the intermediate value theorem to determine zeroes of M_0 and $M_{\frac{1}{2}}$. Finally, by our bounds on the error of M_0 and $M_{\frac{1}{2}}$ in comparison to $E_0(-\frac{1}{2} + iy)$ and $E_{\frac{1}{2}}(\frac{1}{2} + iy)$, we prove that each of the zeroes found in the last step correspond to a zero of $E_0(-\frac{1}{2} + iy)$ and $E_{\frac{1}{2}}(\frac{1}{2} + iy)$.

2. Previous Results

This paper follows methods from [5], which we now summarize. Rankin and Swinnerton-Dyer showed that all the zeroes of the even weight k Eisentstein Series $E_k(z)$ of $SL_2(\mathbb{Z})$ lie on the bottom arc of the standard fundamental domain for $SL_2(\mathbb{Z})\backslash\mathbb{H}$. Let $k = 12n + s$ where $s \in \{0, 4, 6, 8, 10, 14\}$ and

$$(9) \qquad F_k(\theta) = e^{\frac{ik\theta}{2}} E_k(e^{i\theta}) = \frac{1}{2} \sum_{(c,d)=1} (ce^{\frac{i\theta}{2}} + de^{-\frac{i\theta}{2}})^{-k}.$$

Note that $F_k(\theta)$ converges and is real–valued for $0 < \theta < \pi$. Pulling out the largest terms (those with $c^2 + d^2 = 1$), they simplify $F_k(\theta)$ to $F_k(\theta) = 2\cos\left(\frac{k\theta}{2}\right) + R_1$, where R_1 is the sum of the remaining terms. Rankin and Swinnerton-Dyer show that $|R_1| < 2$. Let $\theta_m = \frac{2m\pi}{k}$ where m is an integer such that $\frac{k}{4} \leq m \leq \frac{k}{3}$. As the sign of $F_k(\theta_m)$ changes with each increase of m, the intermediate value theorem can be applied to show a zero exists between each value of m. Using the restrictions on m and θ_m, n zeroes are found. By the valence formula for E_k, it is known that there are at most n zeroes. Thus, all of the zeroes have been accounted for.

Reitzes, Vulakh, and Young [7] further expanded on these methods. They studied the zeroes of the weight $k + \ell$ cusp forms $\Delta_{k,\ell} = E_k E_\ell - E_{k+\ell}$. In this case, all of the zeroes lie on the boundaries of the standard fundamental domain (including the vertical boundaries, unlike in [5]). They begin by using the same method as [5], however they find that their approximations are valid for only $y \ll \sqrt{\frac{k}{\log k}}$, which is the same limitation that we encounter in this paper. For simplicity, they choose to use this approximation for $y \leq k^{\frac{2}{5}}$. As some zeroes lie above this y value, they must develop two other techniques. The first uses the Fourier expansion, rather than the $cz + d$ expansion, to approximate $\Delta_{k,\ell}$ for $y \geq k^{\frac{3}{5}}$. The second technique approximates $\Delta_{k,\ell}$ for $k^{\frac{2}{5}} \leq y \leq k^{\frac{3}{5}}$ by including more terms in their original $cz + d$ approximation.

Though we do not use the techniques from [7], we note them as a possibility for future research; it seems likely that these methods could be used to study larger y values in order to prove Conjecture 1.1 in full, however, the details become quite intricate.

3. Fundamental Domains

In this section we determine fundamental domains for E_∞, E_0, and $E_{\frac{1}{2}}$.

The fundamental domain, F_∞, of $\Gamma_0(4)$ is the union of six regions [3], as shown in Figure 1. These regions are defined as follows:

$$Y = \{z \in \mathbb{H} : |\mathrm{Re}(z)| \leq \tfrac{1}{2}, |z| \geq 1\},$$
$$P = \{z \in \mathbb{H} : |z| < 1, |z + 1| \geq 1, |z - 1| \geq 1\},$$

$$O = \{z \in \mathbb{H} : \mathrm{Re}(z) \geq -\tfrac{1}{2}, |z+1| < 1, |z + \tfrac{1}{3}| \geq \tfrac{1}{3}\},$$
$$G = \{z \in \mathbb{H} : \mathrm{Re}(z) \leq \tfrac{1}{2}, |z - \tfrac{1}{3}| \geq \tfrac{1}{3}, |z - 1| < 1\},$$
$$B = \{z \in \mathbb{H} : |z - \tfrac{1}{3}| < \tfrac{1}{3}, |z - \tfrac{1}{5}| \geq \tfrac{1}{5}, |z - \tfrac{2}{3}| \geq \tfrac{1}{3}\},$$
$$N = \{z \in \mathbb{H} : \mathrm{Re}(z) \leq \tfrac{1}{2}, |z - \tfrac{2}{3}| < \tfrac{1}{3}, |z - \tfrac{3}{8}| \geq \tfrac{1}{8}\}.$$

Note that $F_\infty = Y \cup P \cup O \cup G \cup B \cup N$ is not quite a true fundamental domain, as it is possible for two distinct points in F_∞ to be $\Gamma_0(4)$-equivalent if and only if both points lie on the outer boundaries. However, for our purposes this definition is sufficient.

Let $S(z) = -\frac{1}{z}$ and $T(z) = z + 1$. The boundaries of F_0, the fundamental domain that corresponds to F_∞ for $E_0(z)$, (shown in Figure 2) can be found via $z \mapsto \frac{1}{4}S(z)$. Similarly, the boundaries of $F_{\frac{1}{2}}$, the fundamental domain that corresponds to F_∞ for $E_{\frac{1}{2}}(z)$, (also shown in Figure 2) can be found via $z \mapsto -ST^2S(z) + \frac{1}{2}$.

4. Approximations of E_0

4.1. Properties of $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy)$.

Proposition 4.1. *The function $e^{\frac{\pi i k}{4}} E_0(-\frac{1}{2} + iy)$ is real–valued.*

Proof. To prove this, we use the Fourier expansion given in (3) instead of the $uz+v$ expansion given in (2). Letting $z = -\frac{1}{2} + iy$, we have $q^\ell = (-e^{-2\pi y})^\ell$, which is real–valued. Thus, we need to prove that $e^{\frac{\pi i k}{4}} b_\ell$ is real for all ℓ. We do this in two cases.

Case 1: ℓ squarefree.

When ℓ is squarefree, [4] (see p. 189) simplifies b_ℓ to

$$b_\ell = \frac{\pi^{\frac{k}{2}} \ell^{\frac{k}{2} - 1}}{\Gamma(\frac{k}{2}) e^{\frac{\pi i k}{4}}} \sum_{\substack{n_0 > 0, \\ \text{odd, and} \\ \text{squarefree}}} \sum_{\substack{n_1 | \ell \\ n_1 \text{odd}}} \epsilon_{n_0 n_1^2}^{k+1} (n_0 n_1^2)^{-\frac{k}{2}} \left(\frac{-\ell}{n_0}\right) \sqrt{n_0} \mu(n_1) n_1,$$

where $\mu(n_1)$ is the Möbius function. Note that $\epsilon_{n_0 n_1^2}^{k+1} = \pm 1$ as k is odd. Thus, $e^{\frac{\pi i k}{4}} b_\ell \in \mathbb{R}$ for all ℓ squarefree.

Case 2: ℓ not squarefree.

Let $\ell = p^{2v} \ell_0$ and $p^2 \nmid \ell_0$. By [4] (see pp.195-199),

$$\frac{b_\ell}{b_{\ell_0}} = \begin{cases} 2^{(k-2)v}, & p = 2 \\ \sum_{h=0}^{v} p^{h(k-2)}, & p \text{ odd prime } p \mid \ell_0 \\ \sum_{h=0}^{v} p^{h(k-2)} - \chi_{(-1)^\lambda \ell_0}(p) p^{\lambda - 1} \sum_{h=0}^{v} p^{h(k-2)}, & p \text{ odd prime } p \nmid \ell_0. \end{cases}$$

where $\lambda = \frac{k-1}{2}$ and $\chi_{(-1)^\lambda \ell_0} = \left(\frac{-1}{p}\right)^\lambda \left(\frac{\ell_0}{p}\right)$. Thus, $\frac{b_\ell}{b_{\ell_0}} \in \mathbb{R}$. By iterating this process, we see that $\frac{b_\ell}{b_{\ell_*}} \in \mathbb{R}$ where ℓ_* is squarefree. Furthermore, $e^{\frac{\pi i k}{4}} b_\ell = C e^{\frac{\pi i k}{4}} b_{\ell_*}$ where $C \in \mathbb{R}$. Thus $e^{\frac{\pi i k}{4}} b_\ell$ is real–valued for all ℓ. □

Now we return to the $uz + v$ expansion of $E_0(z)$ given in Equation (2); in the next two subsections, we aim to find a trigonometric approximation for this infinite sum that is accurate for k large enough. Thus, consider the following terms of Equation (2), which are the largest terms when $x = -\frac{1}{2}$. Letting $u = 1$ and $v = 0$, we get the term $\frac{1}{z^{\frac{k}{2}}}$. Letting $u = 1$ and $v = 1$, we get the term $\frac{1}{(z+1)^{\frac{k}{2}}}$. Though it is not obvious at this point, we want to multiply our approximation by $e^{\frac{\pi i k}{4}} r^{\frac{k}{2}}$ where $r = |-\frac{1}{2} + iy|$. Thus, let

$$(10) \qquad M_0(z) = e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} \left(\frac{1}{z^{\frac{k}{2}}} + \frac{1}{(z+1)^{\frac{k}{2}}} \right).$$

The following two sums include all of the terms left to be bounded. Again, we want to multiply the sums by $e^{\frac{\pi i k}{4}} r^{\frac{k}{2}}$, giving

$$(11) \qquad J_1 = e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} \sum_{v \neq 0, 1} \frac{(\frac{-v}{1}) \epsilon_1^k}{(z+v)^{\frac{k}{2}}}, \qquad J_2 = e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} \sum_{\substack{(u, 2v)=1 \\ u > 1}} \frac{(\frac{-v}{u}) \epsilon_u^k}{(uz+v)^{\frac{k}{2}}}.$$

By these definitions, $e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} E_0(z) = M_0(z) + J_1 + J_2$.

Proposition 4.2. *Let $\delta = \arctan(2y)$ and consider $\frac{1}{2} \leq y \leq \frac{\sqrt{k}}{\sqrt{2 \log k}}$. For k large,*

$$M_0(-\tfrac{1}{2} + iy) = \begin{cases} 2\cos\left(\frac{\delta k}{2} - \frac{\pi}{4}\right), & k \equiv 1 \ mod \ 4 \\ 2\cos\left(\frac{\delta k}{2} + \frac{\pi}{4}\right), & k \equiv 3 \ mod \ 4, \end{cases}$$

$|J_1| = o(1)$, *and* $|J_2| \ll \left(\frac{8}{81}\right)^{\frac{k}{4}}$.

Proof. The proof is given in the following two subsections.

4.2. **Simplifying $M_0(-\frac{1}{2} + iy)$.** We convert to polar notation by letting $\frac{1}{2} + iy = re^{i\delta}$ where $\delta = \arctan(2y)$. Note that this implies $-\frac{1}{2} + iy = re^{i(\pi - \delta)}$. Thus,

$$(12) \qquad M_0(-\tfrac{1}{2} + iy) = e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} \left(\frac{1}{(re^{i\delta})^{\frac{k}{2}}} + \frac{i^k}{(re^{i(\pi-\delta)})^{\frac{k}{2}}} \right) = e^{\frac{\pi i k}{4}} \left(e^{\frac{-i\delta k}{2}} + i^k e^{\frac{-i(\pi-\delta)k}{2}} \right).$$

As $e^{ix} = \cos(x) + i\sin(x)$, this becomes

$$(13) \qquad M_0(-\tfrac{1}{2} + iy) = e^{\frac{\pi i k}{4}} \left[\cos\left(\frac{-\delta k}{2}\right) + i\sin\left(\frac{-\delta k}{2}\right) + \cos\left(\frac{-k\pi}{2} + \frac{k\delta}{2}\right) + i\sin\left(\frac{-k\pi}{2} + \frac{k\delta}{2}\right) \right].$$

When $k \equiv 1 \bmod 4$, this can be further simplified by the the even/odd properties of cosine and sine as well as angle identities $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$ and $\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$. Then apply the identity $\cos(x) + \sin(x) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ to obtain

$$(14) \qquad M_0(-\tfrac{1}{2} + iy) = 2\cos\left(\frac{\delta k}{2} - \frac{\pi}{4}\right).$$

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Similarly, when $k \equiv 3 \mod 4$,

$$(15) \qquad M_0(-\tfrac{1}{2} + iy) = 2\cos\left(\tfrac{\delta k}{2} + \tfrac{\pi}{4}\right).$$

Thus,

$$M_0(-\tfrac{1}{2} + iy) = \begin{cases} 2\cos\left(\tfrac{\delta k}{2} - \tfrac{\pi}{4}\right), & k \equiv 1 \mod 4 \\ 2\cos\left(\tfrac{\delta k}{2} + \tfrac{\pi}{4}\right), & k \equiv 3 \mod 4. \end{cases}$$

Note that $M_0(-\tfrac{1}{2} + iy)$ is real–valued. This will be used in Section 6.

4.3. **Bound on Remaining Terms of E_0.** Next, we wish to show that $|J_1| = o(1)$ and $|J_2| \ll \left(\tfrac{8}{81}\right)^{\frac{k}{4}}$ for k large and $\tfrac{1}{2} \leq y \leq \tfrac{\sqrt{k}}{\sqrt{2\log k}}$.

We begin by bounding J_1. By the triangle inequality,

$$(16) \qquad |J_1| \leq \sum_{v \neq 0,1} \frac{\left|r^{\frac{k}{2}} e^{\frac{\pi i k}{4}} \left(\tfrac{-v}{1}\right)\epsilon_1^k\right|}{|z+v|^{\frac{k}{2}}} = \sum_{v \neq 0,1} \frac{r^{\frac{k}{2}}}{|z+v|^{\frac{k}{2}}}.$$

When $x = -\tfrac{1}{2}$, $|z+v| = |z-v+1|$, giving

$$(17) \qquad |J_1| \leq 2\sum_{v \geq 2} \frac{r^{\frac{k}{2}}}{|z+v|^{\frac{k}{2}}} = 2\left[\frac{r^{\frac{k}{2}}}{|\tfrac{3}{2}+iy|^{\frac{k}{2}}} + \sum_{v \geq 3} \frac{r^{\frac{k}{2}}}{|z+v|^{\frac{k}{2}}}\right],$$

by pulling out the term $v = 2$. As $\frac{1}{|z+v|^{\frac{k}{2}}}$ is decreasing for $v \geq 0$, we can bound the sum by an integral using the integral test, giving

$$(18) \qquad |J_1| \leq 2r^{\frac{k}{2}}\left[\frac{1}{\left(\tfrac{9}{4}+y^2\right)^{\frac{k}{4}}} + \int_2^\infty \frac{dt}{(t-\tfrac{1}{2})^2+y^2} \frac{1}{\left((t-\tfrac{1}{2})^2+y^2\right)^{\frac{k}{4}-1}}\right].$$

Utilizing the bound on t and pulling out factors that do not depend on t gives

$$(19) \qquad |J_1| \leq 2r^{\frac{k}{2}}\left[\frac{1}{\left(\tfrac{9}{4}+y^2\right)^{\frac{k}{4}}} + \frac{1}{\left(\tfrac{9}{4}+y^2\right)^{\frac{k}{4}-1}} \int_2^\infty \frac{dt}{(t-\tfrac{1}{2})^2+y^2}\right].$$

This integral can be bounded by an absolute constant, giving

$$(20) \qquad |J_1| \ll 2\left(\frac{r^2}{\tfrac{9}{4}+y^2}\right)^{\frac{k}{4}} + 2r^2\left(\frac{r^2}{\tfrac{9}{4}+y^2}\right)^{\frac{k}{4}-1}.$$

Replacing r by its definition, $r = |\tfrac{1}{2}+iy| = \sqrt{\tfrac{1}{4}+y^2}$, we get

$$(21) \qquad |J_1| \ll \left(\tfrac{1}{4}+y^2\right)\left(\frac{\tfrac{1}{4}+y^2}{\tfrac{9}{4}+y^2}\right)^{\frac{k}{4}-1},$$

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We now simplify the right factor, which we denote as R, precisely

$$(22) \qquad R = \Big(\frac{\frac{1}{4} + y^2}{\frac{9}{4} + y^2}\Big)^{\frac{k-4}{4}} = \Big(\frac{1 + \frac{1}{4y^2}}{1 + \frac{9}{4y^2}}\Big)^{\frac{k-4}{4}}.$$

Note that R is increasing for all $y \geq 0$. Thus, we only need to bound it for large y, specifically $y = \frac{\sqrt{k}}{\sqrt{2\log k}}$. To begin, we convert this to an exponential function and utilize the definition of the logarithm of a quotient. Thus,

$$(23) \qquad R = \exp\Big(\frac{k-4}{4}\log\Big(\frac{1 + \frac{1}{4y^2}}{1 + \frac{9}{4y^2}}\Big)\Big).$$

Using the Taylor expansion of $\log(1+x)$ and simplifying gives

$$(24) \qquad R = \exp\Big(-\frac{k}{2y^2} + \frac{2}{y^2} + O(\frac{k}{y^4}) + O(\frac{1}{y^4})\Big) \ll \exp\Big(-\frac{k}{2y^2} + O(\frac{k}{y^4}) + O(\frac{1}{y^4})\Big).$$

Substitute this into (21) and let $y \leq \frac{\sqrt{k}}{\sqrt{2\log k}}$ to get

$$(25) \qquad |J_1| \ll \Big(\frac{1}{4} + \frac{2k}{\log k}\Big)\exp\Big(-\frac{\log k}{4} + O(\frac{k}{y^4}) + O(\frac{1}{y^4})\Big).$$

The error terms become unnecessary for k large, implying that

$$(26) \qquad |J_1| \ll \Big(\frac{2k}{\log k}\Big)\exp\Big(-\frac{\log k}{4}\Big) = \Big(\frac{2k}{\log k}\Big)k^{-\frac{1}{4}} = o(1).$$

Thus $|J_1| = o(1)$ for k large and $y \leq \frac{\sqrt{k}}{\sqrt{2\log k}}$.

Next we bound J_2, which is simpler. By the triangle inequality,

$$(27) \qquad |J_2| \leq \sum_{(u,2v)=1,u>1} \frac{|r^{\frac{k}{2}} e^{\frac{\pi i k}{4}} (\frac{-v}{u})\epsilon_u^k|}{|uz + v|^{\frac{k}{2}}} \leq \sum_{u\geq 3} r^{\frac{k}{2}} \sum_{v\in\mathbb{Z}} \frac{1}{((v+ux)^2 + u^2y^2)^{\frac{k}{2}}}..$$

Consider the inner sum, which we denote as $S(v)$. Let $f(v) = \frac{1}{((v+ux)^2 + u^2y^2)^{\frac{k}{2}}}$ and note that f is nonnegative, initially increasing, and eventually decreasing. By Equation (3.6) in [7], these properties imply that $\sum_{v\in\mathbb{Z}} f(v) \leq \max f(v) + \int_{-\infty}^{\infty} f(t)dt$. Thus,

$$(28) \qquad S(v) \leq f(-ux) + \int_{-\infty}^{\infty} \frac{dt}{((t+ux)^2 + u^2y^2)^{\frac{k}{2}}} = \frac{1}{(u^2y^2)^{\frac{k}{2}}} + \int_{-\infty}^{\infty} \frac{dt}{(t^2 + u^2y^2)^{\frac{k}{2}}},$$

as $+ux$ was merely a shift. Let $t = uyw$ for some w and note that $dt = uydw$. This gives

$$(29) \qquad S(v) \le \frac{1}{(uy)^k} + \frac{uy}{(uy)^k} \int_{-\infty}^{\infty} \frac{dw}{(1+w^2)^{\frac{k}{2}}}.$$

We can evaluate this integral in terms of $\Gamma-$functions using a table or Mathematica. From here, Stirling's Formula implies that this integral equals $O(k^{\frac{-1}{2}})$. For more details, refer to integral I_k on page 14 of [7]. This implies that

$$(30) \qquad S(v) \ll \frac{1}{(uy)^k} + \frac{1}{(uy)^{k-1}} \frac{1}{\sqrt{k}}.$$

Substituting this into our outer sum, we find that

$$(31) \qquad |J_2| \ll \sum_{u \ge 3} r^{\frac{k}{2}} \left[\frac{1}{(uy)^k} + \frac{1}{(uy)^{k-1}} \frac{1}{\sqrt{k}} \right] \ll r^{\frac{k}{2}} \left[\frac{1}{(3y)^k} + \frac{1}{(3y)^{k-1}} \frac{1}{\sqrt{k}} \right].$$

Simplifying gives

$$(32) \qquad |J_2| \ll \left(\frac{\sqrt{\frac{1}{4}+y^2}}{9y^2} \right)^{\frac{k}{2}} + \left(\frac{\frac{1}{4}+y^2}{9y^2} \right)^{\frac{k}{2}} \frac{3y}{\sqrt{k}} \ll \left(\frac{\sqrt{\frac{1}{4}+y^2}}{9y^2} \right)^{\frac{k}{2}} \left(\frac{3y}{\sqrt{k}} \right).$$

As the first factor of this is decreasing in y, we can bound it by the value at y=1/2.

$$(33) \qquad |J_2| \ll \left(\frac{1}{324y^4} + \frac{1}{81y^2} \right)^{\frac{k}{4}} \left(\frac{3y}{\sqrt{k}} \right) \le \left(\frac{8}{81} \right)^{\frac{k}{4}} \left(\frac{3y}{\sqrt{k}} \right),$$

as $y \ge \frac{1}{2}$. Assuming $y \le o(\sqrt{k})$, this gives

$$(34) \qquad |J_2| \ll \left(\frac{8}{81} \right)^{\frac{k}{4}}. \qquad \qquad \square$$

Note that, as $e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} E_0(z) = M_0 + J_1 + J_2$, and $e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} E_0(z), M_0 \in \mathbb{R}$, it must be the case that $(J_1 + J_2) \in \mathbb{R}$. This will be used in Section 6.

5. Approximations of $E_{\frac{1}{2}}$

This section is similar to Section 4, except that we work with $E_{\frac{1}{2}}(\frac{1}{2} + iy)$ instead of $E_0(-\frac{1}{2} + iy)$.

5.1. Properties of $E_{\frac{1}{2}}(\frac{1}{2} + iy)$.

Proposition 5.1. *The function $E_{\frac{1}{2}}(\frac{1}{2} + iy)$ is real–valued.*

Proof. Recall the automorphy relation (8) from Section 1. Letting $z = -\frac{1}{2} + iy$, this relation becomes

(35) $$E_{\frac{1}{2}}(-\tfrac{1}{2} + iy) = (2iy)^{-\frac{k}{2}} E_0(\tfrac{1}{2} + iy'),$$

for some $y' \in \mathbb{R}$. Utilizing the periodicity of E_0, the right side changes to

(36) $$L := e^{\frac{\pi i k}{4}} (2iy)^{\frac{k}{2}} E_{\frac{1}{2}}(-\tfrac{1}{2} + iy) = e^{\frac{\pi i k}{4}} E_0(-\tfrac{1}{2} + iy').$$

From Proposition 4.1, the right side is real–valued and thus L must also be. Utilizing the automorphy relation (7) from Section 1 and simplifying gives

(37) $$L = i^k e^{\frac{\pi i k}{4}} (2y e^{\frac{\pi i}{2}})^{\frac{k}{2}} E_{\frac{1}{2}}(\tfrac{1}{2} + iy) = (-1)^k (2y)^{\frac{k}{2}} E_{\frac{1}{2}}(\tfrac{1}{2} + iy).$$

As noted earlier, L is real–valued, so $E_{\frac{1}{2}}(z)$ must real–valued when $x = \frac{1}{2}$. $\qquad\square$

In the next proposition, we aim to find a trigonometric approximation for $E_{\frac{1}{2}}(\frac{1}{2} + iy)$, as defined in Equation (2), that is accurate for k large enough. We use methods similar to that of Section 4. Thus, consider the following terms, which are the largest terms when $x = \frac{1}{2}$. Letting $c = 0$ and $d = 1$, we get the term $\frac{1}{z^{\frac{k}{2}}}$. Letting $c = -1$ and $d = 1$, we get the term $\frac{1}{(z-1)^{\frac{k}{2}}}$. Similar to the case of $E_0(z)$, we want to multiply our approximation by $r^{\frac{k}{2}}$ where $r = |\frac{1}{2} + iy|$. Thus, let

(38) $$M_{\frac{1}{2}}(z) = r^{\frac{k}{2}}\left(\frac{1}{z^{\frac{k}{2}}} + \frac{1}{(z-1)^{\frac{k}{2}}}\right).$$

The following two sums include all of the terms left to be bounded. Again, we want to multiply the sums by $r^{\frac{k}{2}}$, giving

(39) $$N_1 = \sum_{c \neq 0,1} \frac{r^{\frac{k}{2}} e^{-\frac{\pi i k}{4}} G(\frac{1-2c}{8})^k}{(z+c)^{\frac{k}{2}}}, \qquad N_2 = \sum_{\substack{(d,2c)=1 \\ d>1}} \frac{r^{\frac{k}{2}} e^{-\frac{\pi i k}{4}} G(\frac{d-2c}{8d})^k}{(dz+c)^{\frac{k}{2}}}.$$

By these definitions, $r^{\frac{k}{2}} E_{\frac{1}{2}}(z) = M_{\frac{1}{2}}(z) + N_1 + N_2$.

Proposition 5.2. *Let* $\theta = \arctan(2y)$ *and consider* $\frac{1}{2} \leq y \leq \frac{\sqrt{k}}{\sqrt{2\log k}}$. *For* k *large,* $M_{\frac{1}{2}}(\frac{1}{2} + iy) = 2\cos\left(\frac{\theta k}{2}\right)$, $|N_1| = o(1)$, *and* $|N_2| \ll \left(\frac{8}{81}\right)^{\frac{k}{4}}$.

Proof. Using methods similar to those in Subsection 4.2, it can easily be shown that $M_{\frac{1}{2}}(\frac{1}{2} + iy) = 2\cos\left(\frac{\theta k}{2}\right)$.

Using techniques similar to those in Subsection 4.3, the bounds on $|N_1|$ and $|N_2|$ quickly become identical to the bounds of $|J_1|$ and $|J_2|$, respectively. Thus, $|N_1| = o(1)$ and $|N_2| \ll \left(\frac{8}{81}\right)^{\frac{k}{4}}$.

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Note that $M_{\frac{1}{2}}(\frac{1}{2}+iy) \in \mathbb{R}$. Furthermore, as $r^{\frac{k}{2}}E_{\frac{1}{2}}(z) = M_{\frac{1}{2}}+N_1+N_2$, and $r^{\frac{k}{2}}E_{\frac{1}{2}}(z), M_{\frac{1}{2}} \in \mathbb{R}$, it must be the case that $(N_1 + N_2) \in \mathbb{R}$. These properties will be used in Section 6. $\square$

6. Sample Points

6.1. **Sample Points for $M_0(-\frac{1}{2}+iy)$ and $M_{\frac{1}{2}}(\frac{1}{2}+iy)$.** In this subsection, we find sample points of $M_0(-\frac{1}{2}+iy)$ and $M_{\frac{1}{2}}(\frac{1}{2}+iy)$ with alternating signs. This will allow us to use the intermediate value theorem, which indicates the number of zeroes we have found for each function. These zeroes are then shown to represent zeroes of $E_0(z)$ and $E_{\frac{1}{2}}(z)$ in the next subsection.

Recall that $M_0(-\frac{1}{2}+iy)$ is a real–valued function (where $\delta = \arctan 2y$). Note that, as $M_0(-\frac{1}{2}+iy)$ approximates $e^{\frac{\pi ik}{4}}E_0(-\frac{1}{2}+iy)$ for $\frac{1}{2} \leq y \leq \frac{\sqrt{k}}{\sqrt{2\log k}}$, we can restrict δ to the interval $\frac{\pi}{4} \leq \delta \leq \arctan\left(\frac{\sqrt{2k}}{\sqrt{\log k}}\right)$. From here on, we use the notation $y_{\max} = \frac{\sqrt{2k}}{\sqrt{\log k}}$.

We wish to find sample points of this function that have the greatest absolute value. Thus, for $k \equiv 1 \mod 4$, we want $\frac{\delta k}{2} - \frac{\pi}{4} = n\pi$ for some $n \in \mathbb{N}$. Solving for δ, we find $\delta = \frac{2\pi n}{k} + \frac{\pi}{2k}$. Substituting this into our interval for δ above, we get $\frac{\pi}{4} \leq \frac{2\pi n}{k} + \frac{\pi}{2k} \leq \arctan(y_{\max})$. Next we solve for n, getting

$$(40) \qquad \frac{k}{8} - \frac{1}{4} \leq n \leq \frac{k}{2\pi}\arctan(y_{\max}) - \frac{1}{4}.$$

Let us simplify the upper bound. By the identity $\arctan x = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)$ and the Taylor expansion of $\arctan(\frac{1}{x})$, we have

$$(41) \qquad n \leq \frac{k-1}{4} - O(\frac{k}{y_{\max}}).$$

As the sign of $\cos\left(\frac{\delta k}{2} - \frac{\pi}{4}\right)$ changes every time n increases, this describes at least $\frac{k}{8} - O(\frac{k}{y_{\max}})$ sign changes. By the intermediate value theorem, there must be a zero of $M_0(-\frac{1}{2}+iy)$ between each of these sign changes, so the number of zeroes that we have found is greater than or equal to $\frac{k}{8} - O(\frac{k}{y_{\max}})$. We can use the same process for $k \equiv 3 \mod 4$, finding that $\frac{k}{8} + \frac{1}{4} \leq n \leq \frac{k+1}{4} + O(\frac{k}{y_{\max}})$. This implies the number of zeroes that we have found is again greater than or equal to $\frac{k}{8} - O(\frac{k}{y_{\max}})$.

Next, recall that $M_{\frac{1}{2}}(\frac{1}{2}+iy) = 2\cos\left(\frac{\theta k}{2}\right)$ is a real–valued function that is valid for $\frac{1}{2} \leq y \leq y_{\max}$. Again, by the same process we find that $\frac{k}{8} \leq n \leq \frac{k}{4} - O(\frac{k}{y_{\max}})$. Again, this implies the number of zeroes that we have found is greater than or equal to $\frac{k}{8} - O(\frac{k}{y_{\max}})$.

6.2. **Main Theorem.** We now prove Theorem 1.2.

Proof. First, we prove that each zero of $M_0(-\frac{1}{2}+iy)$ corresponds to a zero of $E_0(-\frac{1}{2}+iy)$. Recall Proposition 4.2. By our bounds, $|J_1 + J_2| < 2 = |M_0(-\frac{1}{2}+iy)|$ at the sample

points, so each sign change of $M_0(-\frac{1}{2} + iy)$ found above also corresponds to a sign change of $e^{\frac{\pi i k}{4}} r^{\frac{k}{2}} E_0(-\frac{1}{2} + iy)$. Therefore, by the intermediate value theorem, we have found greater than or equal to $\frac{k}{8} - O(\frac{k}{y_{\max}})$ zeroes of $E_0(-\frac{1}{2} + iy)$ for k large. Similarly, by Proposition 5.2, we have found greater than or equal to $\frac{k}{8} - O(\frac{k}{y_{\max}})$ zeroes of $E_{\frac{1}{2}}(\frac{1}{2} + iy)$ when k is large. $\qquad \square$

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