Integral Metapletic Modular Categories

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1. Background Information
   - What is Topological Quantum Computation?
   - What are Integral Metapletic Modular Categories?

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4. Link Invariants associated with these Categories

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- Effects such as superposition, entanglement
- Potential for exponential speedup compared to classical computers on certain applications
- Challenges: Required conditions, decoherence

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- Anyon systems modeled using modular categories

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- This corresponds to all anyon types being observable

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A category corresponds to a braid group representation of finite image (*Property F*) if and only if it is weakly integral (all objects have dimension d such that $d^2 \in \mathbb{Z}$)

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Our work focuses on verifying the property F conjecture for integral metaplectic modular categories

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- We showed that they are group theoretical, which implies property F
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- This means that using these anyon systems, we can't create a universal quantum computer using braiding alone

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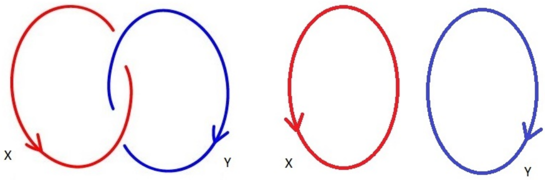

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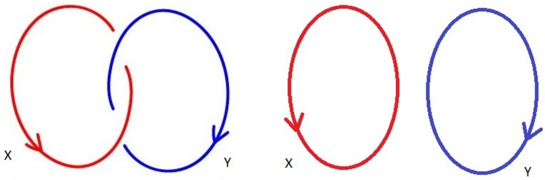

Definition

For any subcategory $\mathcal{L}$ of a braided fusion category $\mathcal{C}$ the *centralizer* of $\mathcal{L}$ denoted by $\mathcal{Z}_{\mathcal{C}}(\mathcal{L})$ is the subcategory consisting of objects $Y \in \mathcal{C}$ that centralize all objects $X \in \mathcal{L}$

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A modular category $\mathcal{C}$ is group theoretical if and only if it is integral and there is a symmetric subcategory $\mathcal{L}$ such that $(\mathcal{Z}_{\mathcal{C}}(\mathcal{L}))_{ad} \subset \mathcal{L}$.

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The unitary modular category $SO(N)_2$ for odd N > 1 has two simple objects, X_1, X_2 of dimension $\sqrt{N}$, two simple objects $\mathbf{1}$, Z of dimension 1, and $\frac{N-1}{2}$ objects $Y_i, i = 1, ..., \frac{N-1}{2}$ of dimension 2.

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The fusion rules are:

1. $Z \otimes Y_i \cong Y_i, Z \otimes X_{it} \cong X_i \pmod 2$, $Z^{\otimes 2} \cong \mathbf{1}$
2. $X_i^{\otimes 2} \cong \mathbf{1} \oplus \bigoplus_i Y_i$,
3. $X_1 \otimes X_2 \cong Z \oplus \bigoplus_i Y_i$,
4. $Y_i \otimes Y_j \cong Y_{min\{i+j, N-i-j\}} \oplus Y_{|i-j|}$, for $i \neq j$ and
   $Y_i^{\otimes 2} = \mathbf{1} \oplus Z \oplus Y_{min\{2i, N-2i\}}$

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Recall, $\mathcal{C} = \{\mathbf{1}, Z, Y_1, Y_2, ..., Y_i, X_1, X_2\}$ for $i = 1, ..., \frac{N-1}{2}$

Lemma

Every integral metaplectic modular category $\mathcal{C}$ with the fusion rules of $SO(N)_2$ for odd N has a symmetric subcategory $\mathcal{L}$ generated by $\mathbf{1}, Z$ and Y_{it} where $t = \sqrt{N}$ and $1 \leq i \leq \frac{t-1}{2}$.

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- $Y_{it} \otimes Y_{jt} \cong Y_{min\{(i+j)t,(t-i-j)t\}} \oplus Y_{|(i-j)t|}$
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Therefore, $\mathcal{L}$ is a fusion subcategory of $\mathcal{C}$

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- Thus, $2t(dim(\mathcal{Z}_{\mathcal{C}}(\mathcal{L})) = 4t^2$ and $dim(\mathcal{Z}_{\mathcal{C}}(\mathcal{L})) = 2t$.

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a *G-grading* is a partitioning of a category $\mathcal{D}$ such that the parts are indexed by elements of G and if $X \in \mathcal{D}_g$, $Y \in \mathcal{D}_h$ then $X \otimes Y \in \mathcal{D}_{gh}$

There is a faithful $\mathbb{Z}_2$ grading on $\mathcal{C}$:

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As $C_{pt} \subset \mathcal{L}$, this means $\mathcal{Z}_{\mathcal{C}}(\mathcal{L}) \subset \mathcal{C}_{\mathbf{1}}$ and we only need to examine $\mathcal{C}_{\mathbf{1}}$.

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- preserves the fusion rules, i.e., if $X \otimes Y = A \oplus B$ then $F[X] \otimes F[Y] = F[A] \oplus F[B]$

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$$\begin{gathered} \mathbf{1} \rightsquigarrow 0 \\ \mathrm{Z} \rightsquigarrow 0 \\ \mathrm{Y}_i \rightsquigarrow i \oplus -i \in \mathbb{Z}_{t^2} \end{gathered}$$

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The trivial component of the de-equivariantization $\mathcal{D}_0$ preserves braiding. $\mathcal{D}_0$ is the image of $\mathcal{C}_{\mathbf{1}}$—exactly what we need.

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We know if $|\langle i \rangle| = t$, this corresponds to a subcategory of dimension $2t$ containing exactly $\frac{t-1}{2}$ distinct Y_i, $\mathbf{1}$ and Z.

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In the previous proof we saw that $\mathcal{Z}_{\mathcal{C}}(\mathcal{L}) = \mathcal{L}$. Therefore, $(\mathcal{Z}_{\mathcal{C}}(\mathcal{L}))_{ad} = \mathcal{L}_{ad}$, so clearly $(\mathcal{Z}_{\mathcal{C}}(\mathcal{L}))_{ad} \subset \mathcal{L}$ and $\mathcal{C}$ is group theoretical.

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- $g \otimes X_a \simeq Y_{\frac{k-1}{2}-a}$, and $g^2 \otimes X_a \simeq X_a$, and $g^2 \otimes Y_a \simeq Y_a$ for $1 \leq a \leq (k-1)/2$
- $X_a \otimes X_a = 1 \oplus g^2 \oplus X_{min\{2a, k-2a\}}$
- $X_a \otimes X_b = X_{min\{a+b, k-a-b\}} \oplus X_{|a-b|}$ when $(a \neq b)$
- $V_1 \otimes V_1 = g \oplus \bigoplus_{a=1}^{\frac{k-1}{2}} Y_a$
- $gV_1 = V_3, gV_3 = V_4, gV_2 = V_1, gV_4 = V_2$ and $g^3 V_a = V_a^*, V_2 = V_1^*, V_4 = V_3^*$

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- $Y_i \otimes Y_j \cong Y_{min\{i+j, 2k-i-j\}} \oplus Y_{|i-j|}$, when $i + j \neq k$
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Gradings

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$$\begin{aligned} \mathcal{C}_{\mathbf{1}} &= \{\mathbf{1}, g^2, Y_i\} \text{ where } i \text{ is even} \\ \mathcal{C}_g &= \{V_1, V_4\} \\ \mathcal{C}_{g^2} &= \{g, g^3, Y_i\} \text{ where } i \text{ is odd} \\ \mathcal{C}_{g^3} &= \{V_2, V_3\}. \end{aligned}$$

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. So, clearly $(\mathcal{Z}_{\mathcal{C}}(\mathcal{L}))_{ad} \subset \mathcal{L}$ and $\mathcal{C}$ is group theoretical.

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Recall, $\mathcal{C} = \{\mathbf{1}, f, g, fg, Y_0, \dots Y_{\frac{k}{2}-1}, X_1, \dots X_{\frac{k}{2}-2}, V_1, V_2, W_1, W_2\}$

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The fusion rules are:

- $f^{\otimes 2} = g^{\otimes 2} = \mathbf{1}, f \otimes X_i = g \otimes X_i = X_{r-i-1}$ and $f \otimes Y_i = g \otimes Y_i = Y_{r-i}$
- $g \otimes V_1 = V_2, f \otimes V_1 = V_1$ and $f \otimes W_1 = W_2, g \otimes W_1 = W_1$
- $V_1^{\otimes 2} = \mathbf{1} \oplus f \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1^{\otimes 2} = \mathbf{1} \oplus g \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1 \otimes V_1 = \bigoplus_{i=0}^{r} Y_i$

$$X_i \otimes X_j = \begin{cases} X_{i+j+1} \oplus X_{j-i-1} & i < j \leq \frac{r-1}{2} \\ \mathbf{1} \oplus fg \oplus X_{2i+1} & \text{i = j ¡ } \frac{r-1}{2} \\ \mathbf{1} \oplus f \oplus g \oplus fg & \text{i = j = } \frac{r-1}{2} < r-1 \end{cases}$$

$$Y_i \otimes Y_j = \begin{cases} X_{i+j} \oplus X_{j-i-1} & i < j \leq \frac{r}{2} \\ \mathbf{1} \oplus fg \oplus X_{2i} & \text{i = j < } \frac{r-1}{2} \\ \mathbf{1} \oplus f \oplus g \oplus fg & \text{i = j = } \frac{r}{2} \end{cases}$$

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- $Y_i^{\otimes 2} \cong \mathbf{1} \oplus f \oplus g \oplus fg$, when $i = \frac{k}{2}$
- $Y_i^{\otimes 2} \cong \mathbf{1} \oplus fg \oplus Y_{min\{2i, 2k-2i\}}$, when $i \neq \frac{k}{2}$
- $Y_i \otimes Y_j \cong Y_{min\{i+j, 2k-i-j\}} \oplus Y_{|i-j|}$, when $i + j \neq k$
- $Y_i \otimes Y_j \cong g \oplus f \oplus Y_{|i-j|}$, when $i + j = k$.

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$$\begin{aligned} \mathcal{C}_{\mathbf{1}} &= \{\mathbf{1}, f, g, fg, Y_i\} \text{ where } i \text{ is even} \\ \mathcal{C}_g &= \{V_1, V_2\} \\ \mathcal{C}_f &= \{W_1, W_2\} \\ \mathcal{C}_{fg} &= \{Y_i\} \text{ where } i \text{ is odd.} \end{aligned}$$

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. So, $(\mathcal{Z}_{\mathcal{C}}(\mathcal{L}))_{ad} \subset \mathcal{L}$ and $\mathcal{C}$ is group theoretical.

Good News!!

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But first, any questions?

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- We can get candidate groups by computing their doubles' ranks with GAP!

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- Data already available for doubles of groups of order <47

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- Thanks to Angus Gruen's honors thesis, we know that $SO(8)_2$ comes from SmallGroup[32,49] (extraspecial group of order 32)

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  Subcategory structure of $\text{Rep}D^{\omega}(G)$ is also known [NNW]

Link Invariants

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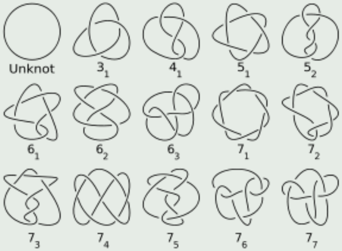
Example (Table of Knots)



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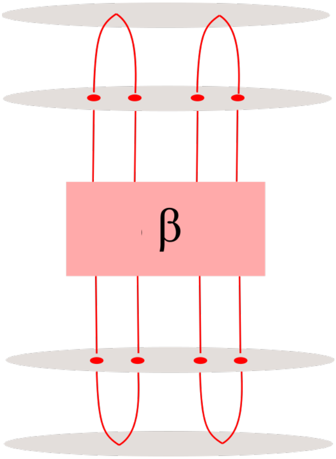

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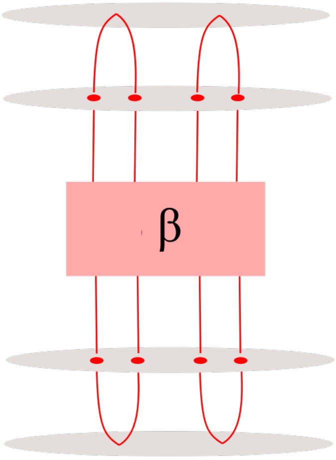

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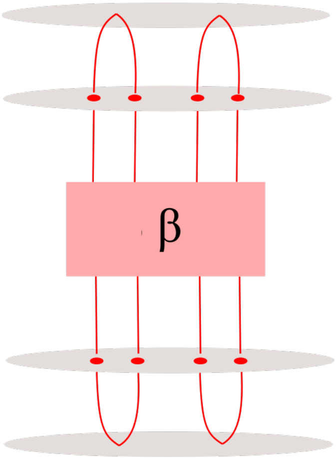

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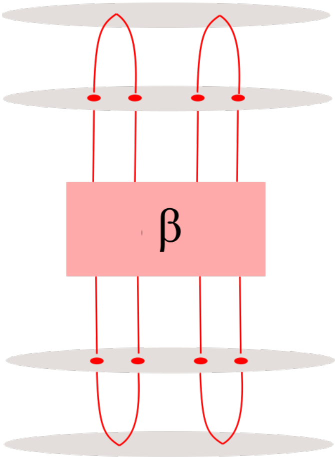

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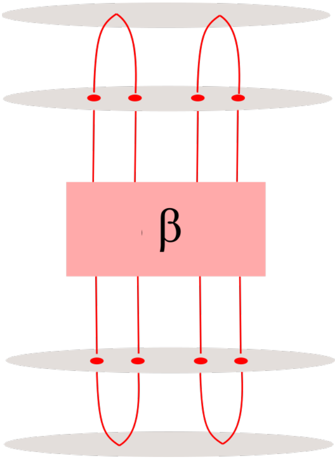

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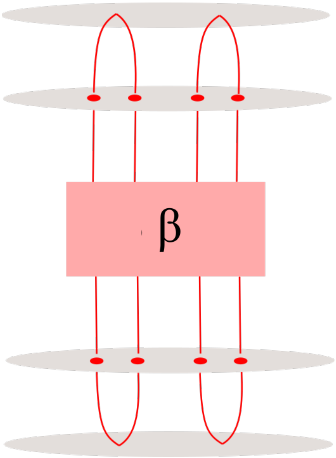

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- This is equivalent to evaluating $Inv(\hat{\beta})$[a]

[a] At a point. May distinguish between fewer knots.

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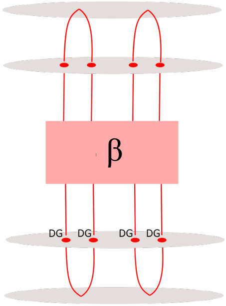
$Inv_{\mathcal{C}}(\hat{\beta}) =$ 

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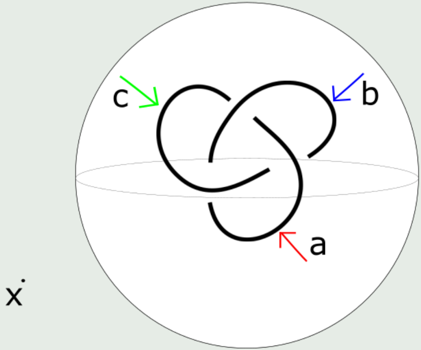

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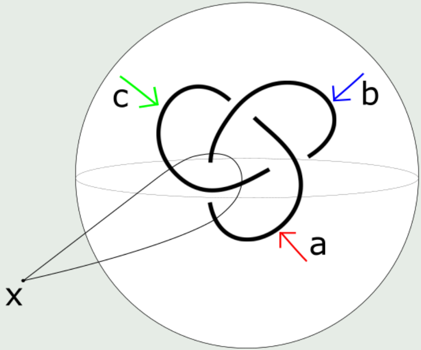

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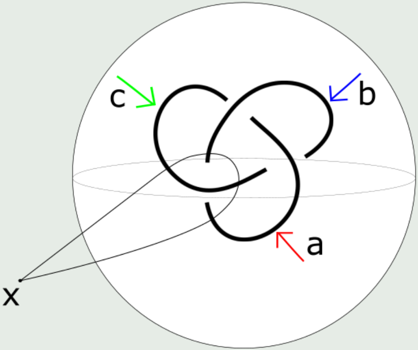

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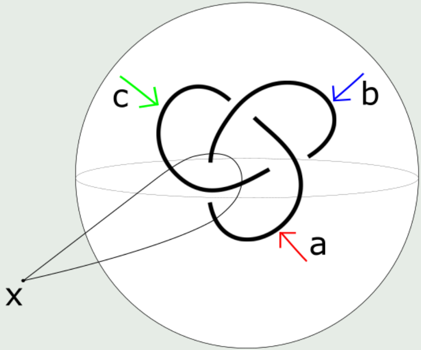

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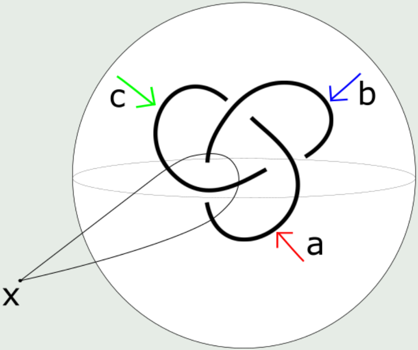

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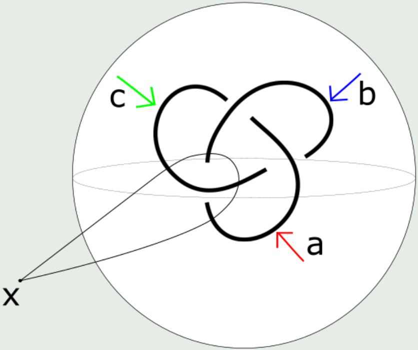

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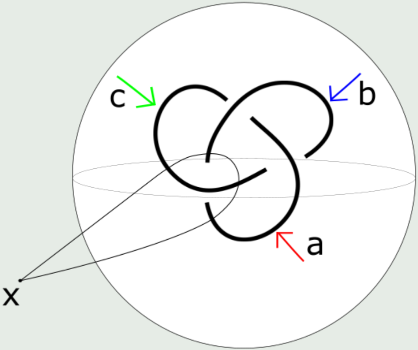

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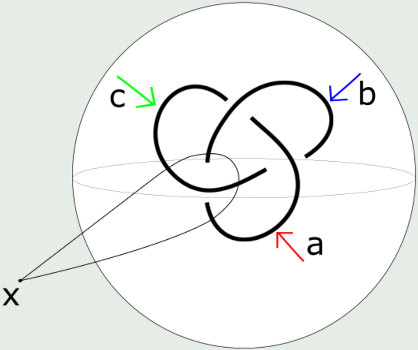

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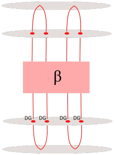

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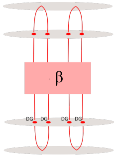

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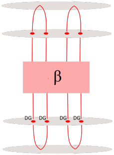
$$Hom(\pi(\mathbf{S}^3 \setminus L, x), G) =$$ 

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. So, we can compute $Inv_{\mathcal{C}}(\hat{\beta})$! But what classical invariant is this?

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The 2-variable *Kauffman polynomial* $K_{q,r}(L)$ is associated with $U_{q,so(n)}$ so the link invariant for our categories (fusion rules of $SO(N)_2$) must be associated with $K_{q,r}(L)$ for some q, r.

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- These categories are especially interesting because of the extra symmetry they have.
- We know that the link invariant associated with these categories is the 2-variable Kauffman polynomial evaluated at $q = e^{\frac{\pi i}{8}}$ $r = -q^{-1}$ [Tuba, Wenzl]

The 2-Variable Kauffman Polynomial (Wenzl's Construction)

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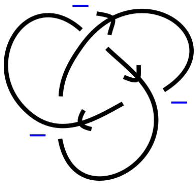
For example, for the trefoil $\omega(L) = -3$



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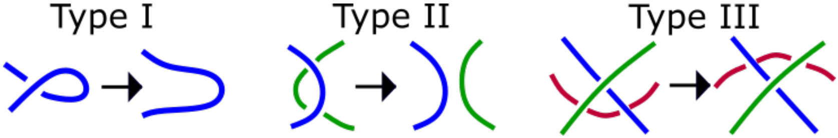

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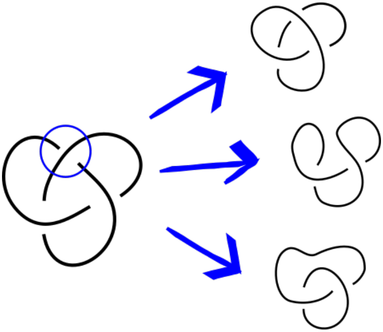

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The 2-Variable Kauffman Polynomial and $SO(8)_2$

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- In particular, we want q and r to be some particular roots of unity. Let $q = e^{\frac{\pi i}{8}}$ $r = -q^{-1}$ [Tuba, Wenzl]

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- The skein relation that we have been using is Wenzl's construction which connects the 2-Variable Kauffman Polynomial to Quantum Groups [Tuba, Wenzl].

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Kauffman's Original Skein Relation:

- $\tilde{K}(\bigcirc) = 1$
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- $\tilde{K}(\diagup\!\!\!\!\diagdown) + \tilde{K}(\diagdown\!\!\!\!\diagup) = z(\tilde{K}()() + \tilde{K}(\asymp))$

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$$K_D(L) = (-1)^{c(L)-1}K(L) \text{ with } a = ir, z = -i(q - q^{-1})$$

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$(-q, q^{-1}+q)$	$\frac{1}{2}(-1)^{c(L)-1}\sum_{X\subset L} q^{4\,\text{linking number}(X, L-X)}$, see [10]
$(-iq^2, q^{-1}+q)$	$[t^{2\lambda(L)}(t^{-\frac{1}{2}}+t^{\frac{1}{2}})(t^{-1}+1+t)^{-1}\sum_{X\subset L}(-1)^{c(X)}V(X^{p(2)})]_{t=-iq^{-1}}$
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$(-q^3, q^{-1}+q)$	$[V(L)]_{t=q^{-4}}$

[Lickorish]

Note: there are no restrictions on q. The q in the table is not the same q that Wenzl used in his version of the Kauffman Polynomial

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- Then, from Lickorish's table we know

$$K(L) = \frac{1}{2}(-1)^{c(L)-1} \sum_{X \subset L} (q^3)^{\text{4linkingnumber(X, L-X)}}$$

Final Results

Combining our mapping and the expression for the original 2-Variable Kauffman Polynomial we know:

Theorem (Mavrakis, Poltoratski, Timmerman, Warren)

The link invariant associated with categories with the fusion rules of $SO(8)_2$ is

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- We don't have to go through the process of using the skein relation to compute the expression for Wenzl's construction of the 2-Variable Kauffman Polynomial and plug in $q = e^{\frac{\pi i}{8}}$ and $r = -q^{-1}$
- We can perform all of our quantum computations for anyons from these categories using this expression

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Thank you!